Calculation method for elastic parabolic cable subjected to uniformly distributed load on each segment and concentrated load at many points

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**ABSTRACT**

Using the assumption that the load is evenly distributed in the horizontal direction, the article has given the cable deflection equation as a function that depends only on the horizontal coordinates, the length of the cable and horizontal distance between two supports. This result leads to the construction of a general system of equations to calculate the deflection, tension, and elongation of an elastic single cable resting on two supports with or without high difference, bearing uniformly distributed loads (or evenly distributed at intervals) and load is concentrated at many points. Calculations of examples to compare with results have been performed by other methods.

**1. Introduction**

Cables are used in many construction projects such as suspension bridges, domes, power transmission lines, cable car systems transporting people and goods, etc. Computational models of cables are also developed to suit the properties of each type of building in which it is applied and mainly develop in two directions:

1) The first direction: Using the finite molecule method, divide the cable into small segments, each cable segment is an element and the split points are nodes. The simplest element most commonly used in the analysis of cable structures is the two-node straight bar element having only axial stiffness (Ozdemir 1979). Elastic catenary cable formulation was first presented by O’Brien and Francis (1964) and Jayaraman and Knudson (1981), it requires less number of elements in cable structures modeling (Andreu et al. 2006, Yang and Tsay 2007). The parabolic elastic element is also used due to its simpler shape compared to catenary (Irvine 1992).

2) The second direction: Based on the assumption that the load is uniformly distributed in the horizontal direction, the deflection equation of the cable span has a parabolic shape.
A cable suspended horizontally under the influence of its own weight has the shape of a catenary. The mathematical solution of the catenary is attributed to James Bernoulli in 1691. In 1794, again in connection with the design of a proposed suspension bridge it was found that, if the cable’s weight was assumed to be uniformly distributed along the span rather than along the cable, the cable hung in a parabolic profile. Ren et al. (2008), presented a finite element (FE) analysis of two-node parabolic cable element for analysis of cable structures with considering static behavior. They concluded that their results are in good agreement with the results obtained from theory of parabolic cable that considers the nonlinear effects. Rezaiee-Pajand et al. (2018) proposed a novel element for considering nonlinear thermo-elastic effects in cables. Abed et al. (2013) also analyzed the cable structures using nonlinear assumption for general loading cases. Some researchers implemented multiple-node (e.g. four and six node) isogeometric elements for analyzing and modeling of cables (Coyette & Guisset, 1988; Ali & Abdel-Ghaffar, 1995; Jian-hua & Wen-zhang, 2015; Wang et al. 2013). However, such models are relatively complex and with a large number of degrees of freedom. Several scholars and researchers have also modeled the cables subjected to different applied loadings (Jiang et al. 2022; Impollonia et al. 2011; Castro-Fresno et al.; 2008; Greco et al. 2014).

In some research works the parabola method has been utilized for the design, analysis and calculation of strength or deformation of cables (Ren et al. 2008; Tibret, 1999; Wang & Yang 1996). The parabolic cable has since received considerable attention, not only because of its simplicity, but also because in many situations (such as suspension bridges), a substantial part of the load is uniformly distributed along the span.

A inextensible cable hanging under its own weight, if the ratio of sag to span is 1:8, or less, the load may be assumed uniformly distributed along the span (Fig.1a), then the deflection equation will have the form of: \[ y = \frac{4f_c}{d} \left( x - \frac{x^2}{d} \right) . \] For this cable, the horizontal component of cable tension is: \[ H = \frac{qd^2}{8f_c} \] and the longitudinal tension at any point in the cable is: \[ T = H \sqrt{1 + \left[y'(x)^2\right]} \] ; where \( q \) is the weight of the cable per unit length, \( d \) is the cable span distance, \( f_c \) is the cable deflection at mid-span \( \left( x = \frac{d}{2} \right) \). Also the length of the cable is determined from: \[ L = \int_0^d \sqrt{1 + \left[ \frac{4f_c}{d} \left( 1 - \frac{2x}{d} \right) \right]^2} \, dx . \] The concept of cable elasticity received little attention until 1858 when Rankine gave an approximate solution for the increase in sag obtained when an inextensible, free-hanging parabolic cable is allowed to stretch. Then, it was not until 1891 that Routh gave the solution of the elastic catenary. However, Rankine's solution contains unnecessary approximations, while Routh's solution is inconvenient on account of the coordinate system used.

In 1974, Irvine (1974) came up with a solution to solve the limitations of the above two approaches. By calculating the value of the ratio \( \frac{f_c}{d} \) in terms of the approximate value of the length \( L \) over the polynomial expansion, Irvine gave the additional deflection (due to the elasticity of the cable) found to be \[ v = \frac{\Delta H}{H - \Delta H} \frac{qd^2}{2H} \left[ x - \left( \frac{x}{d} \right)^2 \right], \] where \( \Delta H \) is the amount of reduction of the horizontal component of the cable tension.

In this paper, an approach is proposed for the calculation method of elastic parabolic cables, which are subjected to many concentrated loads and at the same time are subjected to uniform loads over each interval. Unlike Irvine, from the exact calculation of the parameter \( u = \frac{4f_c}{d} \) value according to the cable length, \( L \); leads to the cable deflection equation is determined when knowing the distance between the two supports, \( d \), and the parameter \( u \).

In the case of a cable span with a high difference between the two supports, \( h \), with the idea of considering the span of the cable as part of a hypothetical cable span with two supports with no difference in height, it is easy to get the deflection equation of the cable span is expression depends on the parameters \( u, d, h \).

This result is used to build a calculation method for elastic cable spans that are subjected to many concentrated loads and at the same time are subjected to uniform loads on each cable segment. In this case, each cable segment can be considered as a cable span characterized by: \( h, d \) are the height difference, the horizontal distance of the two supports and the parameter \( u \), respectively. From the condition of force balance at the junctions between spans, the system of nonlinear equations of
variables: \( u, h, d \) is obtained. By solving this system of equations according to the Newton-Raphson method the values of quantities of the tensile force along the string, \( T \); horizontal tension component, \( H \); displacements at the loading points are then calculated.

Thus, by the calculation method for the cases of elastic cable spans under uniformly distributed loads, the article builds a general system of equations to calculate the deflection for elastic cable spans subjected to many concentrated loads and at the same time are subjected to uniform loads on each segment, resting on two supports with or without a height difference.

2. Equation of deflection of a single cable

2.1. Single cable with uniform load

![Fig. 1. Model of single cable with uniformly distributed load, resting on two supports: (a) no high difference, (b) high difference \( h \).](image)

Single cable of length \( L(m) \) rests on supports \( O \) and \( A \) with horizontal distance \( d(m) \), no high difference (Fig. 1.a) or high difference (Fig.1.b). Select the coordinate system \( Oxy \) with origin \( O \) coincides with a support (in case two supports are of equal height) or origin \( O \) coincides with a support with higher elevation (in case of high difference), the \( Ox \) axis is horizontal and lying. In the vertical plane connecting the two supports the \( Oy \) axis points downward. Assume the load is uniformly distributed in the horizontal direction with the strength \( q(N/m) \). Table 1 also describes the symbols used in the current research.

**Table 1. Symbols of quantities**

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2.1.1 Single cable with uniformly distributed load, resting on two pillows with no height difference

2.1.1.1 Equation of deflection

The cable of length \( L(m) \), resting on two \( O, A \) supports, has no height difference (Fig. 1.a). Assume the cable is subjected to a uniformly distributed load and is inextensible cable.

From the condition that the moment at any point \((x, y)\) on the cable is zero, the force balance in the \( y \) direction (Fig. 1.a) is obtained:


\[-Hy + Vx - \int_{0}^{\frac{x}{2}} g(t) dt = 0\]  

(1)

\[V = \frac{qzd}{2}\]  

(2)

Leads to the deflection equation

\[y = \frac{gx(d - x)}{2H}\]  

(3)

The cable deflection at \(x = \frac{d}{2}\) is the sag and the horizontal component of cable tension is

\[f_c = \frac{qd^2}{8H}; \quad H = \frac{qd^2}{8f_c}\]  

(4)

Substituting (4) in (3) get the deflection equation:

\[y = \frac{4f_c}{d} \left( x - \frac{x^2}{d} \right)\]  

(5)

The deflection equation of a single cable in the form Eq. (5) was also presented by Irvine (1974). However, Eq. (5) also contains the undefined quantity \(f_c\). Symbol \(u = \frac{4f_c}{d}\), the deflection equation Eq. (5) becomes:

\[y = u \left( x - \frac{x^2}{d} \right)\]  

(6)

If point \(M(x, y)\) lies on the cable, the length of the \(OM\) cable will be:

\[L = \int_{0}^{\frac{x}{2}} \sqrt{1 + \left[ y'(t) \right]^2} \, dt = \int_{0}^{\frac{x}{2}} \sqrt{1 + \frac{u^2}{d^2}(2t - d)^2} \, dt\]  

(7)

Integral Eq. (7) will get:

\[L = \frac{d}{4} \left\{ \sqrt{1 + u^2} + \frac{\ln \left( u + \sqrt{1 + u^2} \right)}{u} + B\sqrt{1 + (Bu)^2} + \frac{\ln \left( Bu + \sqrt{1 + (Bu)^2} \right)}{u} \right\}\]  

(8)

where \(B = \frac{2x - d}{d}\). When \(x = d\), then \(B = 1\), resulting in the length of the span's cable segment:

\[L = \frac{d}{2} \left\{ \sqrt{1 + u^2} + \frac{\ln \left( u + \sqrt{1 + u^2} \right)}{u} \right\}\]  

(9)

Symbol \(A = \frac{2L}{d} > 2\), the Eq. (9) is equivalent to:

\[W(u) = \sqrt{1 + u^2} + \frac{\ln \left( u + \sqrt{1 + u^2} \right)}{u} - A = 0\]  

(10)

Symbol \(a = \sqrt{A^2 - 2A} > 0, \quad b = \sqrt{A^2 - 1} > 0\) found that \(W(u) = \frac{1}{u} \left[ \sqrt{1 + u^2} - \frac{\ln \left( u + \sqrt{1 + u^2} \right)}{u} \right] > 0 \forall u > 0\),

\(W(a) < 0\) and \(W(b) > 0\), so equation \(W(u) = 0\) will exist and have a unique solution on \((a, b)\). The finding of the solution \(u\) will be by the method of consecutively bisecting the segment \([a, b]\).
Eq. (6) is the equation representing the deflection of the span of the cable resting on two supports without height difference, assuming the load is uniformly distributed in the horizontal direction and the cable has no stretch. Where $u$ is the solution of equation (10).

2.1.1.2 Horizontal tension and longitudinal tensile force in the cable

The horizontal component of cable tension is

$$H = \frac{qd^2}{8f_e} = \frac{qd}{2u}$$

(11)

The tension at any point $M(x, y)$ in the cable is

$$T = H\sqrt{1 + \left[ y'(x) \right]^2} = \frac{qd}{2u} \sqrt{1 + \frac{u^2}{d^2}(d - 2x)^2}$$

(12)

2.1.1.3 The elongation of cable span

The tensile stress $\sigma(N/m^2)$ of the cable is a function of $x$ as below equation:

$$\sigma(x) = \frac{T(x)}{F} = \frac{qd}{2F \cdot u} \sqrt{1 + \frac{u^2}{d^2}(d - 2x)^2}$$

(13)

As a result, the element of length $\Delta s$ at point $x$ will have elongation length of $\Delta \varepsilon = \frac{\sigma(x)}{E} \Delta s$. Therefore, the cable length has increased by an amount

$$\Delta L = \int_{0}^{d} \frac{\sigma(x)}{E} ds = \frac{qd}{2E \cdot F \cdot u} \sqrt{1 + \frac{u^2}{d^2}(d - 2x)^2} \cdot \sqrt{1 + y'^2} dx = \frac{qd^2}{2E \cdot F \cdot u} \left[ 1 + \frac{u^2}{3} \right]$$

(14)

To get the cable deflection equation including the wire elongation, it is necessary to solve Eq. (10) with the value of $A$ replaced by $\overline{A} = \frac{2(L + \Delta L)}{d}$ to get the solution $\overline{u}$. Then the equation of deflection of the extensible cable, resting on two supports with no difference in height, with a uniformly distributed load in the horizontal direction is obtained from:

$$y = \overline{u} \left( x - \frac{x^2}{d} \right)$$

(15)

2.1.2 Single cable with uniformly distributed load, resting on two supports with high difference

2.1.2.1 Equation of deflection

The cable of length $L(m)$, resting on two $O, A$ supports, has height difference $h_i(m)$ (Fig. 1.b). Assume the cable is subjected to a uniformly distributed load and is inextensible cable. Found that, the $OA$ cable segment is part of the $OAK$ cable span, with support $K$ having the same height as support $A$. The cable span $OAK$ is called assumed cable span. Choose the coordinates whose origin coincides with the support with the higher height. There is the coordinates of the points $A(d, h_i)$ and $K(d, 0)$.

Let $f_1$ be the deflection (at the mid-span point) of the assumed cable span, then according to (5), the deflection equation of the cable span is obtained as

$$y = \frac{4f_1}{d_1} \left( x - \frac{x^2}{d_1} \right)$$

(16)

Since the length of cable segment $OA$ is equal to $L$, there will be:

$$L = \int_{0}^{d} \sqrt{1 + y'^2} dx = \int_{0}^{d} \sqrt{1 + \frac{16f_1^2}{d_1^2}(d_1 - 2x)^2} dx$$

(17)
By defining symbol \( a = d_1 - d \) and integrating Eq. (17) we get:

\[
L = \left( \frac{d+a}{2} \right)^2 \left\{ \frac{2f_c}{(d+a)} \sqrt{1 + \frac{16f_c^2}{(d+a)^2} + \frac{1}{2} \ln \left| \frac{4f_c}{(d+a)} + \sqrt{1 + \frac{16f_c^2}{(d+a)^2}} \right|} - \frac{2f_c(a-d)}{(d+a)^2} \sqrt{1 + \frac{16f_c^2}{(d+a)^2} + \frac{1}{2} \ln \left| \frac{4f_c}{(d+a)} + \sqrt{1 + \frac{16f_c^2}{(d+a)^2}} \right|} \right\}
\]

(18)

where \( u = \frac{4f_c}{d + a} \), the deflection equation (16) becomes: \( y = u \frac{x(d+a-x)}{(d+a)} \)

(19)

There are:

\[
y(d) = h_s = u, \frac{d(d+a-d)}{(d+a)} = u, \frac{d}{(d+a)}
\]

(20)

\[
a = \frac{h_d d}{ud - h_s}; \quad a - d = \frac{2h_d d - ud^2}{ud - h_s}; \quad a + d = \frac{ud^2}{ud - h_s};
\]

(21)

By assuming symbol \( C(u) = \frac{a - d}{a + d} = \frac{2h_d d - ud^2}{ud^2} = \frac{2h_d}{ud} - 1 \), from Eq. (18) leads to

\[
L = \frac{d}{2(1-C)} \left\{ \sqrt{1 + u^2} + \frac{\ln \left(u + \sqrt{1 + u^2} \right)}{u} - C\sqrt{1 + C^2 u^2} - \frac{\ln \left(Cu + \sqrt{1 + C^2 u^2} \right)}{u} \right\}
\]

(22)

Find the solution \( u \) from the equation:

\[
\Psi(u) = \frac{d}{2(1-C)} \left\{ \sqrt{1 + u^2} + \frac{\ln \left(u + \sqrt{1 + u^2} \right)}{u} - C\sqrt{1 + C^2 u^2} - \frac{\ln \left(Cu + \sqrt{1 + C^2 u^2} \right)}{u} \right\} - L = 0
\]

(23)

Substituting the value \( u \) found in Eq. (19) and Eq. (21), we get the equation of deflection of the inextensible cable \( OA \), resting on two supports with difference in height \( h_s \), with a uniformly distributed load in the horizontal direction as:

\[
y = u \left(x - \frac{x^2}{d}\right) + \frac{h_s x^2}{d^2}
\]

(24)

2.1.2.2 Finding the solution \( u \) of the equation \( \Psi(u) = 0 \)

Symbol \( u_i = \frac{h_i}{d}, \; u_0 = \frac{2h_0}{d}, \; u_2 = \frac{\sqrt{16L^2 - d^2}}{d} \). There is \( 1 - C(u_i) = 2 - \frac{2h_i}{u_id} = 0 \), so \( u = u_i \) is the discontinuity of the function \( \Psi(u) \)

There are \( h_s < 2h_i = \sqrt{4h_i^2} < \sqrt{3h_i^2 + L^2 - d^2} < \sqrt{3L^2 + L^2 - d^2} < \sqrt{4L^2 - d^2} < \sqrt{4L^2 - d^2} \)

(25)

\[
=> \quad u_i = \frac{h_i}{d} < u_0 = \frac{2h_i}{d} < \frac{2}{d} \sqrt{4L^2 - d^2} = \frac{\sqrt{16L^2 - d^2}}{d} = u_2 \quad => \quad u_i < u_0 < u_2
\]

There are \( \lim_{u \to u_i} \Psi(u) = \frac{d}{2} \left[ \frac{1 + 2u_i}{\sqrt{1 + u_i^2}} + \frac{1}{\sqrt{1 + u_i^2}} \right] - L = \frac{d\sqrt{1 + u_i^2} - L = \sqrt{d^2 + h_s^2}}{L} < 0 \)

(26)
Symbol: $\Psi(u) = \begin{cases} 
\Psi(u) 
, \ u > u_i 
\end{cases}$ \hspace{1cm} (27)

$\Rightarrow$ function $\Psi(u)$ is continuous $\forall u \geq u_i$

Considering $u > u_0$: Because $C(u) = \frac{2h_i}{u_d} - 1$

should $-1 < C < 0$ and $1 < 1 - C < 2 \Rightarrow \frac{d}{2(1-C)} \ln \frac{u + \sqrt{1 + u^2}}{C u + \sqrt{1 + C^2 u^2}} - C \sqrt{1 + C^2 u^2} > 0$

$\Rightarrow \Psi(u) = \frac{d}{2(1-C)} \sqrt{1+u'^2} + \frac{d}{2(1-C)} \left[ \ln \frac{u + \sqrt{1+u^2}}{C u + \sqrt{1+C^2 u^2}} - C \sqrt{1+C^2 u^2} \right] - L > \frac{d}{4} \sqrt{1+u'^2} - L$ \hspace{1cm} (28)

Therefore $\Psi(u) > \frac{d}{4} \sqrt{1+u'^2} - L = 0$ \hspace{1cm} (29)

The function $\Psi(u)$ is continuous on $[u_i, u_z]$, satisfying the conditions: $\Psi(u_i) = \sqrt{d^2 + h_i^2} - L < 0$ and $\Psi(u_z) > 0$, so there will exist and only solutions $\Psi(u) = 0$ on $[u_i, u_z]$. The finding the solution $u$ from equation (23) is done by the method of consecutive halving the segment $[u_i, u_z]$.

2.1.2.3 Horizontal component of cable tension and tensile force along the wire

The horizontal component of cable tension $H = qd = q(d + a)$

After substitution $a = \frac{h_i d}{u_d - h_i}$ the following equation is obtained:

$$H = \frac{qd^2}{2(u_d - h_i)}$$ \hspace{1cm} (30)

The tensile force along the cable $T = H \sqrt{1 + \frac{y'^2}{d}}$. From Eq. (24) one gets:

$$y' = u - \frac{2ux}{d} + \frac{2h_x}{d^2} \Rightarrow y'(0) = u, \ y'(d) = \frac{2h_x}{d} - u,$$ from which:

$$T = H \sqrt{1 + \frac{u'^2}{d}} \text{ at high cable support}$$

$$T = H \sqrt{1 + \left( u - \frac{2h_x}{d} \right)^2} \text{ at low cable support}$$ \hspace{1cm} (31)

2.1.2.4 The elongation of cable span

The tensile stress along the cable is a function of $x$ as below equation:

$$\sigma(x) = \frac{T(x)}{F} = \frac{qd^2}{2F(u_d - h_i)} \sqrt{1 + \left( u - \frac{2ux}{d} + \frac{2h_x}{d^2} \right)^2}$$ \hspace{1cm} (32)

from which, the elongation of the cable in the span will be

$$\Delta L = \int_0^L \frac{\sigma(x)}{E} \, ds = \frac{qd^2}{2E.F.(u_d - h_i)} \left[ d + \frac{du^2}{3} + \frac{h_x}{3d} \left( 4h_x - 2ud \right) \right]$$ \hspace{1cm} (33)

If the elongation of the cable is taken into account, it is necessary to solve Eq.(23) with the value of $L$ being replaced by $L = L + \Delta L$ to obtain the solution $\bar{u}$. Then the equation of deflection of the extensible cable, resting on two supports with difference in height $h_s$, load evenly distributed in the horizontal direction:
If the height difference is equal to zero \((h_t = 0)\), the calculation results for the cable span resting on supports with high difference return to the calculated results for the cable span when the supports are not difference in height.

2.2. Single cable under uniformly distributed and concentrated loads

2.2.1 Single cable under uniformly distributed load and only one concentrated load, resting on two supports with high difference

The assumption in this section is that the load is uniformly distributed horizontally with intensity \(q(N/m)\). The load \(Q(N)\) is concentrated at the point \(M(x, y)\) whose height is lower than the two supports (as shown in Fig. 2a) or higher than the low support (Fig. 2b). The \(OA\) cable segment can be considered to be supported by three supports: \(O, M, A\). If the values of parameters \(u_1, u_2\) of the cable segments \((OM, MA)\) and the deflection \(h_1\) at \(M\) can be calculated, then according to section 2.1.2, the deflection equation of the cable segments \(OM\) and \(MA\) will be determined.

\[ y = u \left( x - \frac{x^2}{d} \right) + \frac{h_x x^2}{d^2} \quad (34) \]

Let \(T_1, T_2\) be the longitudinal tension forces of the cable segments \(L_1, L_2\) at the point of applying concentrated load \(M\). On the \(MA\) cable segment, take the \(O_2x_2, y_2\) coordinate system so that the origin of \(O_2\) coincides with the point of higher elevation (with \(A\) in Fig. 2a or with \(M\) in Fig. 2b). The forces \(T_1, T_2\) make up with the horizontal the corresponding angles \(\alpha_1, \alpha_2\).

Let \(d_1 = x, d_2 = d - x\) and \(u_1, u_2\) be the variables in the deflection equation (according to Eq.(24)), respectively the cable segments \(L_1, L_2\). Use Eq.(30) applied to cable segments \(L_1, L_2\) to obtain:

\[ H = \frac{q d_i^2}{2(u_i d_i - h_1)} + \frac{q d_i^2}{2(u_i d_i - h_2)} = \frac{q d_i^2}{2(u_i d_i - h_1)} \quad (35) \]

Due to \(h_2 = |h_1 - h_2| = (h_1 - h_2), sign (h_1 - h_2)\), Eq.(35) is equivalent to:

\[ u_1 d_1^2 + h_1 \left[ sign(h_1 - h_2) d_1^2 + d_2^2 \right] - u_2 d_2^2 - h_2 d_2^2 \cdot sign(h_1 - h_2) = 0 \quad (36) \]

By balancing the forces at \(M\) one gets:

\[ H = T_1 \cos \alpha_1 = T_2 \cos \alpha_2 \quad (37) \]
\[ T_1 \sin \alpha_1 + T_2 \sin \alpha_2 = Q \quad (38) \]
\[ H (\tan \alpha_i + \tan \alpha_z) = Q \]  

(39)

Also from Eq. (24) it can be found:

\[ \tan \alpha_i = \gamma_i'(d_i) = \frac{2h_i}{d_i} - u_i \]  

(40)

\[ \tan(\alpha_z) = \begin{cases} \frac{2h_i}{d_z} - u_z, & h_i > h_z \\ -u_z, & h_i \leq h_z \end{cases} \]  

(41)

or \[ \tan(\alpha_z) = (h_i - h_z) \frac{\text{sign}(h_i - h_z) + 1}{d_z} - u_z \]  

(42)

With the aid of Eq. (35), Eq. (40), Eq. (42), Eq. (39) is more conveniently written as:

\[ h_i \left[ 2d_i d_2 + d_2^2 \left[ \text{sign}(h_i - h_z) + 1 \right] + \frac{2Q}{q} d_2 \right] - u_i \left[ d_i^2 d_2 + \frac{2Q}{q} d_i d_1 \right] - u_z d_i d_2 - d_2^2 \left[ \text{sign}(h_i - h_z) + 1 \right] h_i = 0 \]  

(43)

Applying Eq. (22) to the \( OM \) and \( MA \) cable segments result the below relations:

\[ -4L_i \left[ u_i d_i - h_i \right] + u_i d_i^2 \sqrt{1 + u_i^2} + d_i^2 \ln \left( u_i + \sqrt{1 + u_i^2} \right) + \\
+ \left( u_i d_i - 2h_i \right) \sqrt{d_i^2 + \left( u_i d_i - 2h_i \right)^2} + d_i^2 \ln \left( u_i d_i - 2h_i \right) \sqrt{d_i^2 + \left( u_i d_i - 2h_i \right)^2} = 0 \]  

(44)

\[ -4L_i \left[ u_z d_z - h_z \right] + u_z d_z^2 \sqrt{1 + u_z^2} + d_z^2 \ln \left( u_z + \sqrt{1 + u_z^2} \right) + \\
+ \left[ u_z d_z - 2h_z \right] \sqrt{d_z^2 + \left[ u_z d_z - 2h_z \right]^2} + d_z^2 \ln \left[ u_z d_z - 2h_z \right] \sqrt{d_z^2 + \left[ u_z d_z - 2h_z \right]^2} = 0 \]  

(45)

\[ h_z = \left| h_i - h_z \right| = (h_i - h_z) \text{sign}(h_i - h_z) \]  

(46)

\[ d_i + d_z - d = 0 \]  

(47)

Solve the system of 6 equations: (i.e. Eq. (36), Eq. (43), Eq. (44), Eq. (45), Eq. (46), Eq. (47)) to find the \( d_1, d_2, u_i, u_z, h_i, h_z \) values according to the parameters \( L_i, L_z, d, Q, q, h \). Solving the above system of equations is done by Newton-Raphson method. The initial values \( d_1^{(0)}, d_2^{(0)}, u_i^{(0)}, u_z^{(0)}, h_i^{(0)}, h_z^{(0)} \) are calculated according to the section 2.1.2 applied to the cable segments \( OA, OM, MA \) when there is no load \( Q \).

If the longitudinal extension of the cable is taken into account, then after calculating the values of \( u_i, u_z, h_i, h_z \) from the above equations, Eq. (33) can be applied to calculate the elongation on the cable segments \( OM \) and \( MA \) respectively:

\[ \Delta L_1 = \frac{qd_1^2}{2E.F. \left( u_i d_i - h_i \right)} \left[ d_i + \frac{d_i u_i^2}{3} + \frac{h_i}{3d_i} \left( 4h_i - 2u_i d_i \right) \right] \]  

(48)

\[ \Delta L_2 = \frac{qd_2^2}{2E.F. \left( u_z d_z - h_z \right)} \left[ d_z + \frac{d_z u_z^2}{3} + \frac{h_z}{3d_z} \left( 4h_z - 2u_z d_z \right) \right] \]  

(49)

Resolving the system of six equations (i.e. (36), (43), (44), (45), (46), (47)) with the value \( L_i \) replaced by \( L_i = L_i + \Delta L_i, i = 1, 2 \), get the solution \( d_1, d_2, u_i, u_z, h_i, h_z \). From here, the deflection equations for each cable segment A and B are obtained as described by Eq. (34):
\[ y_i = u_i \left( x_i - \frac{x_i^2}{d_i} \right) + \frac{h_i x_i^2}{d_i^2}, \quad i = 1, 2 \] (50)

Tensile force along the cable is determined from:
\[ T_o = H \sqrt{1 + u_i^2} \quad \text{at } O \]
\[ T_A = H \sqrt{1 + \left( \frac{\text{sign}(h_i - h_j) + 1}{d_i} \right)^2} \quad \text{at } A \] (51)

2.2.2 The elastic single cable subjected to a load evenly distributed over each segment and a concentrated load at many points

Consider an elastic single cable resting on two supports \( O, A \) with the difference in height equal to \( h_i (m) \). At points \( M_i (x_i, y_i) \) on the cable is subjected to concentrated load \( Q_i (N) \) \((i = 1+\ldots+n)\) as shown in Fig. 3. On segments \( M_{i-1}M_i \), load evenly is distributed along the cable with intensity \( \gamma_i (N/m) \). Knowing the coordinates of the \( M_i (x_i, y_i) \) points before placing the load, it is necessary to determine the coordinates of the \( M_i (x_i, y_i) \) points and also the deflection equation of each segment of \( M_{i-1}M_i \) cable after placing the load. Table 2 illustrates the symbols and parameters for the investigated cable.

\[ \text{Fig. 3. Model of single cable subjected to a load evenly distributed over each segment and a concentrated load at many points.} \]

<table>
<thead>
<tr>
<th>Table 2. Symbols of quantities for defining the conditions of a single cable subjected to an evenly distributed load over each segment.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Content</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>The length of the ( M_{i-1}M_i ) cable segment, ( i = 1+\ldots+n+1 )</td>
</tr>
<tr>
<td>The load intensity is uniform on the wire of the ( M_{i-1}M_i ) cable segment.</td>
</tr>
<tr>
<td>The projection of ( M_i ) on ( Ox )</td>
</tr>
<tr>
<td>The length of ( N_{i-1}N_i, i = 1+\ldots+n+1 )</td>
</tr>
<tr>
<td>The deflection at ( M_i, i = 1+\ldots+n+1 )</td>
</tr>
<tr>
<td>Tension force in cable segment ( M_{i-1}M_i ) at ( M_i, i = 1+\ldots+n )</td>
</tr>
<tr>
<td>Tension force in cable segment ( M_{i-1}M_i ) at ( M_i, i = 1+\ldots+n )</td>
</tr>
<tr>
<td>The difference in height between two points ( M_{i-1} ) and ( M_i, i = 1+\ldots+n+1 ).</td>
</tr>
<tr>
<td>Angle formed by ( T_i^p ) with the horizontal, ( i = 1+\ldots+n )</td>
</tr>
<tr>
<td>Angle formed by ( T_i^T ) with the horizontal, ( i = 1+\ldots+n )</td>
</tr>
</tbody>
</table>

where \( M_0 \equiv O; \ M_{n+1} \equiv A \), \( y_0 = 0 \), \( y_{n+1} = h_i \). The assumption is that on each segment \( M_{i-1}M_i \) the load is uniformly distributed horizontally with intensity of \( q_i = \gamma_i \frac{L_i}{d_i} (N/m) \).
Use Eq. (30) applied to cable segments \( L_i \) and \( L_{i+1} \) at points \( M_i \) \( (i = 1, n) \) to obtain:

\[
H = \frac{q_i d_i^2}{2(u_i d_i - h_i)} = \frac{q_{i+1} d_{i+1}^2}{2(u_{i+1} d_{i+1} - h_{i+1})} \quad (52)
\]

Substituting \( h_i = |y_i - y_{i-1}| = (y_i - y_{i-1}) \text{sign}(y_i - y_{i-1}) \) into Eq. (52) will get:

\[
\Phi_i = [u_{i+1} d_{i+1} - (y_i - y_{i-1}) \text{sign}(y_i - y_{i-1})] q_i d_i^2 - [u_i d_i - (y_i - y_{i-1}) \text{sign}(y_i - y_{i-1})] q_{i+1} d_{i+1}^2 = 0 \quad i = 1, n \quad (53)
\]

Vertical and horizontal equilibriums at \( M_i \) lead to below equations:

\[
T_i \cos \alpha_i^t = T_i \cos \alpha_{i+1}^t = H \quad (54)
\]
\[
T_i \sin \alpha_i^t + T_i \sin \alpha_{i+1}^t = Q_i \quad (55)
\]
\[
H \left( \tan \alpha_i^t + \tan \alpha_{i+1}^t \right) = Q_i \quad (56)
\]

On each cable segment \( M_i, M_j \) with length \( L_i \) considered in the local coordinate system \( O_i \xi \eta \) \( (O_i \equiv M_k \), \( k \in \{i-1, i\} \), \( y_i = \min \{y_{i-1}, y_i\} \), the deflection equation in the form of Eq. (24) is obtained as:

\[
\eta_i = u_i \left( \xi_i - \frac{\xi_i^2}{d_i^2} \right) + \frac{h_i \xi_i^2}{d_i^2} \quad (57)
\]

Hence get:

\[
\tan(\alpha_i^t) = \frac{(y_i - y_{i-1}) \text{sign}(y_i - y_{i-1}) + 1}{u_i} \quad (58)
\]
\[
\tan(\alpha_{i+1}^t) = \frac{(y_i - y_{i+1}) \text{sign}(y_i - y_{i+1}) + 1}{u_{i+1}} \quad (59)
\]

With the aid of Eq. (52), the Eq.6 is written as:

\[
\Psi_i = d_i^2 \left( \tan \alpha_i^t + \tan \alpha_{i+1}^t \right) - 2 \frac{Q_i}{d_i} [u_i d_i - (y_i - y_{i-1}) \text{sign}(y_i - y_{i-1})] = 0 \quad i = 1, n \quad (60)
\]

Applying the Eq. (22) to \( M_i, M_j \) cable segments will result:

\[
W_i = \frac{d_i}{2(1-C_i)} \left\{ \ln \frac{u_i + \sqrt{1+u_i^2}}{u_i} - C_i \sqrt{1 + C_i^2 u_i^2} - \frac{\ln \left(C_i u_i + \sqrt{1 + C_i^2 u_i^2}\right)}{u_i} \right\} - L_i = 0 \quad i = 1, n+1 \quad (61)
\]

and

\[
\Gamma = \sum_{i=1}^{n+1} d_i - d = 0 \quad (62)
\]

where:

\[
C_i = \frac{2(y_i - y_{i-1}) \text{sign}(y_i - y_{i-1})}{u_i d_i} - 1
\]

Solve the system of \( 3n + 2 \) equations \((53), (60), (61), (62)\) for \( 3n + 2 \) values of \( y_1, ..., y_n, d_1, ..., d_{n+1} \) and \( u_1, ..., u_{n+1} \). The solution of the above system of equations is done by Newton-Raphson method. The initial values \( y_1^{(0)}, ..., y_n^{(0)}, d_1^{(0)}, ..., d_{n+1}^{(0)}, u_1^{(0)}, ..., u_{n+1}^{(0)} \) calculated according to section 2.1.2 are applied to the \( OA \) and \( M_i, M_j \) cable segments in the absence of the \( Q_i, i = 1, n \) and \( q_i, i = 1, n+1 \) loads.
If the longitudinal extension of the cable is taken into account, then after calculating the values of \( y_1, ..., y_n, d_1, ..., d_{n+1} \) and \( u_1, ..., u_{n+1} \) from the above equations, Eq. (33) is applied to calculate the elongation on the cable segments \( M_{i+1}M_i \ (i = 1, n+1) \) respectively:

\[
\Delta L_i = \frac{q_i d_i^2}{2E.F.\left(u_i d_i - |y_i - y_{i+1}|\right)} \left[ d_i + \frac{d_i u_i^2}{3} + \frac{1}{3d_i} \left(4|y_i - y_{i+1}| - 2u_i d_i\right) \right]
\] (64)

Resolving the system of \( 3n+2 \) equations (53), (60), (61) and (62) with the value \( L_i \) replaced by \( L_i = L_i + \Delta L_i \), \( i = 1, n+1 \), get the solution \( y_i', i = 0, n+1 \) and \( \bar{u}_i, \bar{d}_i, i = 1, n+1 \). From here, get the deflection equations in the local coordinate system \( \xi, \eta \) for each \( M_{i+1}M_i, \ i = 1, n+1 \) cable segment.

\[
\eta_i = \bar{u}_i \left(\frac{\xi}{d_i} - \frac{\xi_i^2}{d_i^2}\right) + \frac{1}{d_i} \left(\bar{y}_i - \bar{y}_{i+1}\right) \left(\frac{\xi}{d_i}\right)^2
\] (65)

After applying the load, the coordinates of the load points \( M_{i,i}, i = 1, n \) is \( \bar{x}_i = \sum_{k=1}^{i} \bar{d}_k \) and with \( \bar{x}_0 = 0, \bar{x}_{n+1} = d \) we have the equation of deflection of the elastic cable \( OA \) in the coordinate system \( O\xi\eta \) with each segment \( \bar{x}_{i-1} \leq x \leq \bar{x}_i, \ i = 1, n+1 \):

\[
y = \begin{cases}
\bar{u}_i \left(\frac{(x-x_{i-1})^2}{d_i}\right) + \frac{1}{d_i} \left(\bar{y}_i - \bar{y}_{i+1}\right) \left(\frac{x-x_{i-1}}{d_i}\right)^2 + \bar{y}_{i+1} & \text{if } \bar{y}_i \geq \bar{y}_{i+1} \\
\bar{u}_i \left(\frac{(x-x_i)^2}{d_i}\right) + \frac{1}{d_i} \left(\bar{y}_i - \bar{y}_{i+1}\right) \left(\frac{x-x_{i-1}}{d_i}\right)^2 + \bar{y}_i & \text{if } \bar{y}_i < \bar{y}_{i+1}
\end{cases}
\] (66)

Tensile force along the cable at two supports are obtained from:

\[
T_o = H\sqrt{1+u_i^2} \quad \text{at } O \\
T_A = H\sqrt{1+u_{n+1}^2} \quad \text{at } A
\] (67)

3. Calculation applications

3.1. Elastic cable suspended with point load

The following application is taken as reference to validate different methods to simulate the cables. These cases were analyzed by some researchers such as: O’Brien and Francis (1964), Jayaraman and Knudson (1981), Tibert (1998) and Andreu et al. (2006). The problem is to determine the displacement of the loading point \( M \), when the pre-stressed cable bears its own weight and is subjected to concentrated loads. The cable has a self-weight: \( q = 46.12 \text{ N/m} \), a cross-sectional area: \( F = 5.484 \text{cm}^2 \) and an elastic modulus: \( E = 13100 \text{kN/cm}^2 \). Initial configuration and further information regarding this example are shown in Fig. 4.
Let $L_0, u_0$ and $L, u$ be the corresponding parameters of the in extension and elastic cable spans in the state of self-load.

According to formulas (9) and (14) one gets:

$$L_0 + \Delta L_0 = \frac{d}{2} \left[ \sqrt{1 + u_0^2} + \frac{\ln \left( u_0 + \sqrt{1 + u_0^2} \right)}{u_0} \right] + \frac{q d^2}{2 E F u_0} \left( 1 + \frac{u_0^2}{3} \right) = L = \frac{d}{2} \left[ \sqrt{1 + u^2} + \ln \left( u + \sqrt{1 + u^2} \right) \right]$$

(68)

where $d = \frac{304.8 m \cdot u = \frac{4 \times 30.48}{304.8}}{4} = 0.4$. Solving equation (68) gets $u_0 = 0.398 \Rightarrow L_0 = 312.666 m$. The cable span, when no load is placed at $M$, has $d_L = 121.92 m$, $h_L = 29.26 m$. When the load is applied at $M$ by using equations (36), (43), (45), (46), (47), the initial values are obtained as: $d_{l1}^{(0)} = 35.586; d_{l2}^{(0)} = 182.88; u_{1l}^{(0)} = u_{2l}^{(0)} = 0.398; h_{l1}^{(0)} = h_{l2}^{(0)} = 29.1164$ via solving the equations with Newton-Raphson method. Get the results: $h_l = h_{l2} = 34.8608 m$, $d_{l1} = 121.053 m$; displacements at point $M$ in the $Ox$ and $Oy$ directions were compared with the results of different researchers, as shown in Table 3 and on the graph of Fig. 5. It is seen from Table 3 that the results of the current research are in agreement with the previous works and researcher showing the acceptable assumptions and calculations of the cable with parabola type element.

Table 3. Comparison of displacements obtained with different works at point $M$.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Element type</th>
<th>Displacements (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Horizontal ($Ox$)</td>
</tr>
<tr>
<td>O’Brien and Francis (1964)</td>
<td>Elastic straight</td>
<td>-0.845</td>
</tr>
<tr>
<td>Jayaraman and Knudson (1981)</td>
<td>Elastic catenary</td>
<td>-0.859</td>
</tr>
<tr>
<td>Tibert (1999)</td>
<td>Elastic parabola</td>
<td>-0.866</td>
</tr>
<tr>
<td>Andreu (2006)</td>
<td>Elastic catenary</td>
<td>-0.860</td>
</tr>
<tr>
<td>Present work</td>
<td>Elastic parabola</td>
<td>-0.866</td>
</tr>
</tbody>
</table>

Fig. 5. Image for elastic parabolic cable span under concentrated load.

(1) - The inextensible cable in the state of self-loading; (2) - The inextensible cable in the state of under concentrated load; (3) - The extensible cable in the state of under concentrated load.

3.2. Elastic cable span subjected to many concentrated loads and at the same time are subjected to uniform loads on each segment

The cable span rests on two supports $O$ and $A$ with horizontal distance $d = 200 m$ and height difference $h = 2 m$. Initially, the cable is pulled with a horizontal tension $H = 33,333 kN$, then concentrated loads $P_1 = 20 kN; P_2 = 15 kN$ are applied at points $M_1, M_2$ together with a uniform load of magnitude $q = 90 kN/m$ on the cable interval $M_1, M_2$ (see Fig. 6). The displacement at points $M_1, M_2$ in vertical and horizontal directions and reactions at two supports $O, A$ were calculated after loading. The cable has a self-weight: $\gamma = 40 N/m$, a cross-sectional area: $F = 8 cm^2$ and an elastic modulus: $E = 13100 kN/cm^2$. 
Let \( L_0 \), \( u_0 \) and \( L_u \) be the corresponding parameters of the in extension and elastic cable spans in the state of self-load. Before the load is applied, the cable span is stretched due to its own weight, with \( d = 200 \text{ m} \), \( h_i = 2 \text{ m} \) and horizontal component of cable pulling force \( H = 33,333 \text{kN} \). According to Eq. (30), get: 
\[
0.13 = \frac{\gamma d}{2H} + \frac{h_i}{d} = 0.13 .
\]
The coordinates of the points are \( M_1 : M_1 (40, 4.24); M_2 (90, 6.84) \). Using the formulas (22) and (33):
\[
L_0 + \Delta L_0 = \frac{d}{2(1-C)} \left\{ \sqrt{1+u_0^2} + \frac{\ln(u_0 + \sqrt{1+u_0^2})}{u_0} - C(1+u_0^2) - \frac{\ln(Cu_0 + \sqrt{1+C^2u_0^2})}{u_0} \right\} + \frac{\gamma d^2}{2E.F.(u_0d - h_i)} \left[ d + \frac{h_i^2}{3d} + \frac{h_i}{3}(4h_i - 2u_0d) \right]
\]
\[
L = \frac{d}{2(1-C)} \left\{ \sqrt{1+u^2} + \frac{\ln(u + \sqrt{1+u^2})}{u} - C(1+u^2) - \frac{\ln(Cu + \sqrt{1+C^2u^2})}{u} \right\} ,
\]
in which \( C = \frac{2h_i}{d} - 1 \).

Solving equation \( L_0 + \Delta L_0 = L \) gets \( u_0 = 0.13 \Rightarrow L_0 = 200.42 \text{ m} \). After applying the load, the set of equations (53), (60), (61), (62) are solved by Newton - Raphson method with initial values: \( u^{(0)}_1 = 0.121 \), \( u^{(0)}_2 = 0.0766 \), \( u^{(0)}_3 = 0.101 \), \( y^{(0)}_1 = 3.952 \), \( y^{(0)}_2 = 6.394 \); \( d^{(0)}_1 = 40 \), \( d^{(0)}_2 = 50 \), \( d^{(0)}_3 = 110 \). The corresponding results are: \( \bar{y}_1 = 7.12 \), \( \bar{y}_2 = 9.43 \), \( \bar{d}_1 = 39.67 \text{ m} \); \( \bar{d}_2 = 50.14 \text{ m} \). Displacements at point \( M_i \) in the \( Ox \) and \( Oy \) directions is shown in Table 4 and on the graph of Fig. 7. The reactions at \( O \) and \( A \) in vertical directions: \( V_0 = 33,152 \text{kN} \); \( V_A = 14,358 \text{kN} \) and in the horizontal direction is: \( H = 180.3 \text{kN} \).

**Table 4. Comparison of displacement point \( M \)**

<table>
<thead>
<tr>
<th>Point</th>
<th>Horizontal (( Ox ))</th>
<th>Vertical (( Oy ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>-0.33</td>
<td>2.88</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>-0.19</td>
<td>2.59</td>
</tr>
</tbody>
</table>

**Fig. 7.** Image for elastic cable span simultaneously bears many concentrated loads and uniform loads on each segment. (1) - The inextensible cable in the state of self-loading; (2) - The inextensible cable in the state of under concentrated load; (3) - The extensible cable in the state of under concentrated load.
4. Conclusion

With the load evenly distributed on the cable being considered to be evenly distributed in the horizontal direction and the introduction of the dimensionless parameter \( u = \frac{4f_c}{d} \), the paper has built a system of non-linear equations, calculated the necessary parameters to get the equation of deflection and tension of the elastic cable resting on two high-displacement bearings, simultaneously bearing many concentrated loads and uniform loads on each cable span. Solving the system of equations is done easily by the Newton-Raphson method. Finally, the example calculation results are compared with the calculated results of other studies as acceptable and demonstrating the validity and accuracy of considering the cable with parabola elastic elements.

References


