Engineering Solid Mechanics 13 (2025) 391-396

Contents lists available at GrowingScience

Engineering Solid Mechanics

homepage: www.GrowingScience.com/esm

Buckling of columns under conditions of non-ideality

Mehmet Pakdemirli[a]*

[a]*Department of Mechanical Engineering, Manisa Celal Bayar University, Manisa, Turkey*

A R T I C L E I N F O

Article history:
Received 20 March 2025
Accepted 2 July 2025
Available online
2 July 2025

Keywords:
Buckling
Columns
Non-ideal supports
Strained parameters method
Approximate analytical solutions

A B S T R A C T

The boundary conditions are idealized in mechanics of materials. However, small imperfections may exist which deviate the conditions from being ideal. The non-ideal boundary conditions are modeled using perturbations and applied to buckling problems of columns. A linear and a nonlinear model are treated separately. The effect of such conditions on the critical loads and buckling shapes are analytically calculated using the strained parameter perturbation method. Depending on the mode numbers and the physical parameters, the critical loads may either decrease or increase. The mode shapes of the buckling are also distorted by the imperfections in the support conditions.

1. Introduction

Beams, columns and plates are important structural elements for carrying loads developed within the structures. They may be supported by a variety of conditions such as simply supported, clamped, free, sliding etc. These structures may be supported from the ends as well as there may be some intermediate supports. Normally, the support conditions are idealized such that for a simply supported condition, the deflection and the moment are zero. In reality this may not be the case always. There may be small gaps, loose pins that allow small deflections. There may also be some sort of friction in the hinged joints producing a small moment. For a built-in support, again there may be some looseness at the support allowing for small deflections and the slope may deviate from zero for that condition. These imperfections in the support conditions are named as non-ideal conditions.

Perturbation theory was employed to formulate non-idealities in the pioneering works of Pakdemirli & Boyacı (2001, 2002, 2003, 2003a). Pakdemirli & Boyacı (2001) considered the free vibrations of a nonlinear beam with stretching effects. Pakdemirli & Boyacı (2002) considered vibrations of axially moving string problem as well as a beam problem to investigate the effects of non-ideality in support conditions. Pakdemirli & Boyacı (2003) treated the intermediate support condition for a nonlinear beam model. The externally excited linear beam model supported from the middle was also investigated (Pakdemirli & Boyacı, 2003a). In those pioneering work, it is shown that imperfections in the support conditions may change the natural frequencies, the amplitudes and the frequency response curves. The stretched damped beam vibrations were studied by Boyacı (2006). By employing an iteration-perturbation method, Eigoli & Ahmadian (2011) showed that the frequency-response curves were altered by the imperfections. The clamped conditions case was investigated for Euler Bernoulli and Timoshenko beams by Lee (2013). The non-ideal boundary condition formulation was applied to micro beam problems also. Sarı & Pakdemirli (2013) treated a slightly curved micro-beam on an elastic foundation. Zhang et al. (2013) also considered the vibration of microbeams with flexible supports. Atcı and Bağdatlı (2017) studied the vibrations of fluid conveying micro-beams.

* Corresponding author.
E-mail addresses: pakdemirli@gmail.com (M. Pakdemirli)

ISSN 2291-8752 (Online) - ISSN 2291-8744 (Print)

doi: 10.5267/j.esm.2025.7.001

For vibrations of composite beams with non-ideal boundary conditions, see Ghadiri & Hosseini (2014) and Ghadiri et al. (2015). For multi-supported axially moving strings of non-ideal supports, see Yurddaş et al. (2013). By employing Laplace and Fourier transforms, the dynamic responses of poro-elastic beams were treated by Fallahzadeh & Shariyat (2015). Bağdatlı and Uslu (2015) considered an integro-differential model of beam vibrations in search of the effects of non-idealities. Pade approximations were used for non-ideal clamped supports of beams (Heryodono & Lee, 2019).

Vibrations of plates were also investigated with a similar approach. Aydoğdu & Ece (2006) considered the static buckling and dynamic vibrational problem of rectangular plates with one-side non-ideally supported. The critical buckling load and natural frequencies may alter with imperfections. In another rectangular plate problem considered by Mohammadi & Gheisary (2009), an analytical solution was presented using the Lindstedt Poincare technique. Najafizadeh et al. (2012) treated vibrations of functionally graded plates with imperfections in the conditions by Levy and Lindstedt-Poincare method. They found that frequency and mode shapes may be affected by the edge conditions. Khalili et al. (2013) studied the buckling for plates supported by Pasternak foundation. They found that buckling load may increase or decrease for deviations from the ideality.

The buckling of columns under imperfections is treated for the first time in this study. Both a linear and a non-linear model are considered. Approximate analytical solutions are presented by employing the strained parameter method, a perturbation technique. The criteria are derived for uniformly valid approximations. The effect of deviations from ideality on the critical buckling loads and buckling shapes are analyzed using the solutions. The physical parameters that may result in increase or decrease of critical buckling loads are the mode numbers, amplitudes of buckling and non-ideal deviation parameters.

2. Linear Buckling

The linear buckling of a column is given by the equation (Beer & Johnston, 1992)

$$\frac{d^2y^*}{dx^{*2}}+\frac{P}{EI}y^*=0 \tag{1}$$

where EI is the flexural rigidity, P is the axial compressive load, x^* is the coordinate along the column and $y^*=y^*(x^*)$ is the lateral displacement at each point x^*. In deriving Eq. (1), the moment at a point x^* in the column is $M=-Py^*$ and the curvature is approximated by $\frac{1}{\rho}\cong\frac{d^2y^*}{dx^{*2}}$ for small slopes. The dimensionless form of the equations were obtained by dividing the spatial variables with the length L of the column

$$x=\frac{x^*}{L},\ y=\frac{y^*}{L} \tag{2}$$

Inserting Eq. (2) into Eq. (1) yields the linear dimensionless buckling model

$$\frac{d^2y}{dx^2}+\frac{PL^2}{EI}y=0. \tag{3}$$

The physical problem now depends on the single dimensionless buckling parameter defined by

$$\lambda^2=\frac{PL^2}{EI} \tag{4}$$

and the dimensionless equation reads

$$\frac{d^2y}{dx^2}+\lambda^2y=0. \tag{5}$$

For simply supported ideal end conditions $y(0)=0$, $y(1)=0$ at both ends. Other support conditions can be obtained by defining an effective length instead of the original column length L. For a fixed-free support condition $L_e=2L$, for a fixed-fixed support condition $L_e=L/2$ and for a fixed-hinged condition $L_e=0.699L\approx0.7L$ (Beer & Johnston, 1992). One may now assume that there are slight deviations from the ideal case and due to some imperfections at the joints, small displacements are allowed with the conditions being

$$y(0)=\varepsilon a,\qquad y(1)=\varepsilon b \tag{6}$$

where ε is a perturbation parameter artificially introduced so that the end displacements are small when $a,b\sim O(1),\varepsilon\ll1$. Although the linear problem (5) with the conditions (6) can be solved exactly, to retrieve the eigenvalue characteristics, a perturbation type of solution will be presented. A perturbation expansion

$$y(x) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2) \quad (7)$$

with the buckling parameter expanded to eliminate secularities

$$\lambda^2 = \lambda_0^2 + \varepsilon\lambda_1 + O(\varepsilon^2) \quad (8)$$

is assumed. The method employed is the Strained Parameters Method (Nayfeh, 1981). Substituting Eq. (7) and Eq. (8) into Eq. (5) and Eq. (6) and separating at each order of approximation yields

$$O(1):\ y_0'' + \lambda_0^2 y_0 = 0,\ y_0(0) = 0,\ y_0(1) = 0 \quad (9)$$
$$O(\varepsilon):\ y_1'' + \lambda_0^2 y_1 = -\lambda_1 y_0, \qquad y_1(0) = a,\ y_1(1) = b \quad (10)$$

The first order solution is

$$y_0(x) = c sin n\pi x, \qquad n = 1,2,3,\ldots \quad (11)$$
$$\lambda_0 = n\pi \quad (12)$$

where the amplitude of the mode shape remains arbitrary. The solution at the next level of approximation is

$$y_1(x) = a cos n\pi x + (b(-1)^n - a)x cos n\pi x \quad (13)$$
$$\lambda_1 = \frac{2n\pi}{c}(b(-1)^n - a) \quad (14)$$

Combining both solutions in Eq. (7) and Eq. (8), the final result is

$$y(x) = c sin n\pi x + \varepsilon(a cos n\pi x + (b(-1)^n - a)x cos n\pi x) \quad (15)$$
$$\lambda^2 = n^2\pi^2 + \varepsilon\frac{2n\pi}{c}(b(-1)^n - a) \quad (16)$$

For an admissible perturbation solution, the corrections should be a small fraction of the unperturbed terms. This requires the conditions

$$\left|\frac{\varepsilon a}{c}\right| \ll 1, \qquad \left|\frac{\varepsilon(b(-1)^n - a)}{c}\right| \ll 1 \quad (17)$$

to be satisfied. The second term in the solution (15) gives the distortion of the mode shapes from the ideal modes. From Eq. (4) and Eq. (16), the critical buckling loads for each mode shapes are

$$P_{cr} = \frac{EI}{L^2}\left(n^2\pi^2 + \varepsilon\frac{2n\pi}{c}(b(-1)^n - a)\right),\ n = 1,2,3,\ldots \quad (18)$$

The first term is the famous Euler Formula for the critical loads (Beer & Johnston, 1992) and the second term represents the deviation from the ideal case. The essential remarks follow

- Non-ideal critical load is dependent on maximum amplitude of the ideal mode shape. However, ideal critical load is amplitude independent.
- For odd mode shapes $P_{cr} = \frac{EI}{L^2}\left(n^2\pi^2 - \varepsilon\frac{2n\pi}{c}(b + a)\right)$ and the critical load decreases for $\frac{a+b}{c} > 0$ and increases for $\frac{a+b}{c} < 0$
- For even mode shapes $P_{cr} = \frac{EI}{L^2}\left(n^2\pi^2 + \varepsilon\frac{2n\pi}{c}(b - a)\right)$ and the critical load decreases for $\frac{b-a}{c} < 0$ and increases for $\frac{b-a}{c} > 0$
- For the special case of one non-ideality, without loss of generality, one may assume $b = 0$ with the critical loads being $P_{cr} = \frac{EI}{L^2}\left(n^2\pi^2 - \varepsilon\frac{2n\pi}{c}a\right)$
- Buckling is an unstable position for a column which should be avoided and the most critical loading corresponds to the lowest value $P_{cr} = \frac{EI}{L^2}\left(\pi^2 - \varepsilon\frac{2\pi}{c}(b + a)\right)$ which occurs when n=1.

In **Fig. 1**, the ideal and non-ideal mode shapes are contrasted with each other for the first mode. **Fig. 2** is a comparison for the second mode shapes.

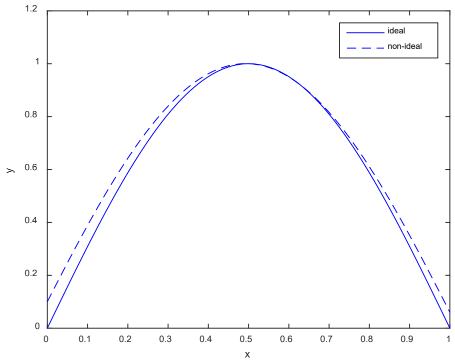

Fig. 1. Deviations of mode shapes (n=1, ε=0.1, a=1, b=0.6, c=1)

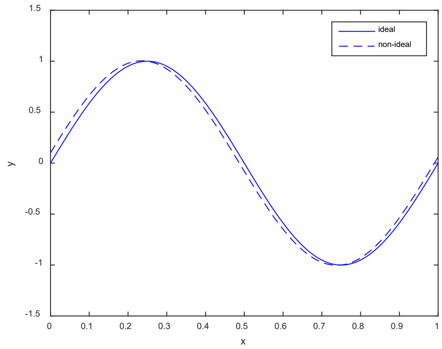

Fig. 2. Deviations of mode shapes (n=2, ε=0.1, a=1, b=0.6, c=1)

There are slight distortions observed in the mode shapes due to the non-ideal supports.

3. Non-Linear Buckling

For the non-linear buckling model, starting from

$$\frac{1}{\rho} = \frac{M}{EI} \tag{19}$$

and substituting for $\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$ and $M = -Py$, the dimensional equation is

$$\frac{d^2y^*}{dx^{*2}} + \frac{P}{EI} y^*(1 + y^{*\prime 2})^{3/2} = 0 \tag{20}$$

The equation is again cast in a non-dimensional form by dividing the spatial variables with the length L of the column

$$x = \frac{x^*}{L}, \ y = \frac{y^*}{L} \tag{21}$$

and substituting into Eq. (20) yielding

$$\frac{d^2y}{dx^2} + \frac{PL^2}{EI} y(1 + y'^2)^{3/2} = 0. \tag{22}$$

In order to delay the geometric nonlinearity and make it appear at the same order as of the non-idealities, a transformation is done $y = \sqrt{\varepsilon} y$ which leads to

$$\frac{d^2y}{dx^2} + \lambda^2 y(1 + \varepsilon y'^2)^{3/2} = 0 \tag{23}$$

in view of Eq. (4). The conditions allowing small deflections are

$$y(0) = \varepsilon a, \qquad y(1) = \varepsilon b \tag{24}$$

where $a, b \sim O(1), \varepsilon \ll 1$ as mentioned earlier.

With employment of the Strained Parameters Method (Nayfeh, 1981), the expansions are

$$y(x) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2) \tag{25}$$
$$\lambda^2 = \lambda_0^2 + \varepsilon \lambda_1 + O(\varepsilon^2) \tag{26}$$

substituted into Eq. (23) and Eq. (24) which yields after separation

$$O(1): \ y_0'' + \lambda_0^2 y_0 = 0, y_0(0) = 0, \ y_0(1) = 0 \tag{27}$$
$$O(\varepsilon): \ y_1'' + \lambda_0^2 y_1 = -\lambda_1 y_0 - \frac{3}{2}\lambda_0^2 y_0 y_0'^2, \quad y_1(0) = a, y_1(1) = b \tag{28}$$

The first order solution is

$$y_0(x) = c\sin n\pi x, \qquad n = 1,2,3,\ldots \tag{29}$$
$$\lambda_0 = n\pi \tag{30}$$

The solution at the next level of approximation is

$$y_1(x) = a\cos n\pi x + (b(-1)^n - a)x\cos n\pi x + \frac{3}{64}n^2\pi^2c^3\sin 3n\pi x \tag{31}$$
$$\lambda_1 = -\frac{3}{8}n^4\pi^4c^2 + \frac{2n\pi}{c}(b(-1)^n - a) \tag{32}$$

The final solutions are

$$y(x) = c\sin n\pi x + \varepsilon\left(a\cos n\pi x + (b(-1)^n - a)x\cos n\pi x + \frac{3}{64}n^2\pi^2c^3\sin 3n\pi x\right) \tag{33}$$
$$\lambda^2 = n^2\pi^2 + \varepsilon\left(\frac{2n\pi}{c}(b(-1)^n - a) - \frac{3}{8}n^4\pi^4c^2\right) \tag{34}$$

For an admissible perturbative solution, the corrections should be a small fraction of the unperturbed terms. This requires the conditions

$$\left|\frac{\varepsilon a}{c}\right| \ll 1, \qquad \left|\frac{\varepsilon(b(-1)^n - a)}{c}\right| \ll 1, \qquad \frac{3}{8}\varepsilon n^2\pi^2c^2 \ll 1 \tag{35}$$

to be satisfied in the expansions. The $O(\varepsilon)$ term in the solution (33) contains distortion of the mode shapes stemming from the combined effect of non-ideal supports and geometric nonlinearity. From Eq. (4) and Eq. (34), the critical buckling loads for each mode shapes are

$$P_{cr} = \frac{EI}{L^2}\left(n^2\pi^2 + \varepsilon\left(\frac{2n\pi}{c}(b(-1)^n - a) - \frac{3}{8}n^4\pi^4c^2\right)\right) \qquad n = 1,2,3,\ldots \tag{36}$$

One may draw the following conclusions from Eq. (36) in addition to the comments given in the previous section for the linear case:

- The geometric nonlinearity always decreases the critical buckling load in accordance with the negative term $-\varepsilon\frac{3}{8}n^4\pi^4c^2$
- The decrease of the critical load due to the geometric nonlinearity can be suppressed by non-ideal support conditions if
  $c = \frac{1}{n\pi}\sqrt[3]{\frac{16}{3}(b(-1)^n - a)}$
- The critical load of the linear theory can be higher, equal or lower for the non-linear, non-ideal case depending on the buckling amplitudes and non-ideal support parameters.

Comparison of the mode shapes with the ideal and non-ideal cases are depicted in **Fig. 3** and **Fig. 4**. The solid curves represent the ideal case with geometric nonlinearity and the dashed curves represent the non-ideal case with geometric nonlinearity

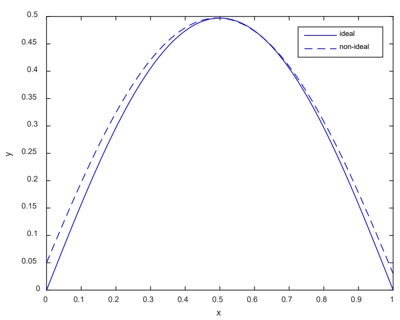

Fig. 3. Deviations of mode shapes (n=1, ε=0.05, a=1, b=0.6, c=0.5)

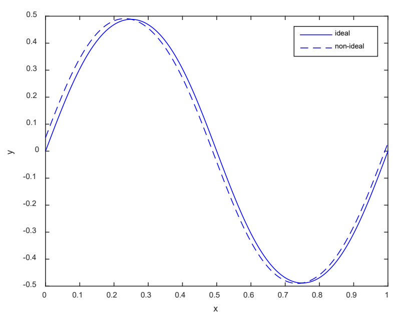

Fig. 4. Deviations of mode shapes (n=2, ε=0.05, a=1, b=0.6, c=0.5)

Non-ideal support conditions distort slightly the mode shapes.

4. Conclusions

The buckling of columns is investigated for the case of small deviations in boundary conditions. The dimensionless equations are solved using the strained parameters method, a perturbation technique well adopted for eigenvalue-eigenfunction problems. Effect of such deviations on the critical buckling loads and mode shapes are investigated in detail. Formulas for the critical buckling loads are given. The non-idealities may change the critical buckling loads depending on the non-ideality parameters, amplitudes of buckling and number of modes. The imperfections in the conditions may suppress the effect of the geometric nonlinearity for some special values of the physical parameters. The mode shapes are also distorted by the non-ideal support conditions.

References

Atcı, D., & Bağdatlı, M. (2017). Vibrations of fluid conveying microbeams under non-ideal boundary conditions. *Microsystems Technologies, 23,* 4741-4752. https://doi.org/10.1007/s00542-016-3255-y.

Aydoğdu, M., & Ece, M. C. (2006). Buckling and vibration of non-ideal simply supported rectangular isotropic plates. *Mechanics Research Communications*, *33*, 532-540. https://doi.org/10.1016/j.mechrescom.2005.08.002.

Bağdatlı, S. M. & Uslu, B. (2015). Free vibration analysis of axially moving beam under non-ideal conditions. *Structural Engineering and Mechanics*, *54*(3), 597-605. https://doi.org/10.12989/sem.2015.54.3.597.

Beer, F. B. & Johnston, E. R. (1992). *Mechanics of Materials*, McGraw Hill Inc., Berkshire, England.

Boyacı, H. (2006). Vibrations of stretched damped beams under non-ideal boundary conditions. *Sadhana*, *31*(1), 1-8. https://doi.org/10.3390/mca6030217.

Eigoli, A. K., & Ahmadian, M. T. (2011). Nonlinear vibration of beams under non-ideal boundary conditions. *Acta Mechanica*, *218*, 259-267. https://doi.org/10.1007/s00707-010-0423-5.

Fallahzadeh, S. R., & Shariyat, M. (2015). Dynamic responses of poroelastic beams with attached mass-spring systems and time dependent, non-ideal supports subject to moving loads: An analytical approach. *JCAMECH*, *46*(2), 133-151.

Ghadiri, M., & Hosseini, M. (2014). Vibration analysis of a composite beam with non-ideal boundary conditions. *International Journal of Basic Sciences and Applied Research*, *3*(2), 103-118.

Ghadiri, M., Malekzadeh, K., & Ghasemi, F. A. (2015). Free vibration analysis of an axially preloaded laminated composite beam carrying a spring-mass-damper system with a non-ideal support. *Jourdan Journal of Mechanical and Industrial Engineering*, *9*(3), 195-207.

Heryodono, A. R. H., & Lee, J. (2019). Free vibration analysis of Euler-Bernoulli beams with non-ideal clamped boundary conditions by using Pade approximation. *Journal of Mechanical Science and Technology*, *33*(3), 1169-1175. https://doi.org/10.1007/s12206-019-0216-2.

Khalili, S. M. R., Abbaspour, P., & Fard, K. M. (2013). Buckling of non-ideal simply supported laminated plate on Pasternak foundation. *Applied Mathematics and Computation*, *219*, 6420-6430. https://doi.org/10.1016/j.amc.2012.12.056.

Lee, J. (2013). Free vibration analysis of beams with non-ideal clamped boundary conditions. *Journal of Mechanical Science and Technology*, 27(2), 297-303. https://doi.org/10.1007/s12206-012-1245-2.

Mohammadi, J., & Gheisary, M. (2009). Effect of non-ideal boundary conditions on buckling of rectangular functionally graded plates. *Journal of Solid Mechanics*, *1*(2), 91-97.

Najafizadeh, M. M., Mohammadi, J., & Khazaeinejad, P. (2012). Vibration characteristics of functionally graded plates with non-ideal boundary conditions. *Mechanics of Advanced Materials and Structures*, *19*(7), 543-550. https://doi.org/10.1080/15376494.2011.563407.

Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques*. John Wiley & Sons, New York, USA.

Pakdemirli, M., & Boyacı H. (2001). Vibrations of a stretched beam with non-ideal boundary conditions", *Mathematical & Computational Applications*, *6*(3), 217-220. https://doi.org/10.3390/mca6030217.

Pakdemirli, M., & Boyacı H. (2002). Effect of non-ideal boundary conditions on the vibrations of continuous systems. *Journal of Sound and Vibration*, *249*(4), 815-823. https://doi.org/10.1006/jsvi.2001.3760.

Pakdemirli, M., & Boyacı H. (2003). Nonlinear vibrations of a simple-simple beam with a non-ideal support in between. *Journal of Sound and Vibration*, *268*(2), 331-341. https://doi.org/10.1016/S0022-460X(03)00363-8.

Pakdemirli, M., & Boyacı H. (2003a). Vibrations of a simply supported beam with a non-ideal support at an intermediate point. *Mathematical & Computational Applications*, *8*(2), 159-164. https://doi.org/10.3390/mca8020159.

Sarı, G., & Pakdemirli, M. (2013). Vibrations of a slightly curved microbeam resting on an elastic foundation with non-ideal boundary conditions. *Mathematical Problems in Engineering*, 2013. https://doi.org/10.1155/2013/736148.

Yurddaş, A., Özkaya, E., & Boyacı, H. (2013). Nonlinear vibrations of axially moving multi-supported strings having non-ideal support conditions. *Nonlinear Dynamics*, *73*, 1223-1244. https://doi.org/10.1007/s11071-012-0650-5.

Zhang, Z. Y., Zhang, W. M., & Meng, G. (2013). Dynamic characteristics of micro-beams considering the effect of flexible supports. *Sensors*, *13,* 15880-15887. https://doi.org/10.3390/s131215880.