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Optimal lot size for an item when holding cost is moved by depletion rate of inventory above a certain stock level

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ABSTRACT

In this paper, a deterministic inventory model with depletion rate dependent holding cost is developed. The demand rate is a power function of the on-hand inventory behind to a certain stock level, at which the demand rate becomes a constant. Shortages are allowed and partially backlogged with the function of waiting time of next replenishment. It is also proved in this model that the optimal replenishment policy not only exists but also is unique. Furthermore, it is provided a simple solution procedure for finding the maximum total profit per unit time. Numerical examples have also been given to illustrate the model and real managerial fact in inventory holding for stock dependent demand.

1. Introduction

In many real-life situations for certain types of consumer goods (i.e. fruits, vegetables, donuts, and others), the consumption rate is sometimes influenced by the stock level. It is usually observed that on sale items more sales and profits are often associated with displaying large piles of consumer goods in a supermarket. The consumption rate may go up or down with the on-hand inventory level respectively. So building up inventory is profitable in this scenario and notion to maintain the cost will play a key role in the stock dependent demand. First of all Levin et al. (1972) investigated the inventory model with stock dependent demand and Baker and Urban (1988), Mandal and Maiti (1997, 1999), Balkhi and Benkherouf (2004) etc. gave a notable contribution in the developing of this concept.

Datta and Pal (1990) modified the model presented by Baker and Urban (1988) by assuming that the stock-dependent demand rate was down to a given level of inventory. By their assumptions, not all customers are attracted to purchase goods by the huge stock. When the stock level declines to a certain stock level, customers arrive to purchase good because of its goodwill, good quality or facilities. The research articles that dealt with stock dependent demand rate are Urban (1992), Pal et al. (1993), Goh (1994), Padmanabhan and Vart (1995), Giri et al. (1996), Roy and Chaudhari (1997), Sarkar et al.

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(1997), Datta and Pal (2001), Chang (2004), Hou and Lin (2006), Min and zhou (2009), Goyal and Chang (2009), Hsieh et al. (2010) and others.

The characteristic of many research articles is that the unsatisfied demand (due to shortages) is completely backlogged. However, in reality, demands for foods, medicines etc. are usually lost during the shortage period. Montgomery et al. (1973) studied both deterministic and stochastic demand inventory models with a mixture of backorder and lost sales. Later, Rasenberg (1979) provided a new analysis of partial backorders. Park (1982) reformulated the cost function and established the solution. Mak (1987) modified the model by incorporating a uniform replenishment rate to determine the optimum production inventory-control policies. For fashionable, commodities and high-tech product with short product life cycle, the willingness for a customer to wait for backlogging during a shortage period is diminishing with the length of the waiting time. Hence, the longer the waiting time, the smaller the backlogging rate. To reflect this phenomenon, Chang and Dye (1999) developed an inventory model in which the proportion of the customers who would like to accept backlogging is the reciprocal of a linear function of the waiting time. Currently, Papachristos and Skouri (2000) established a partially backlogged inventory model in which the backlogging rate decreases exponentially as the waiting time increase. Teng et al. (2002, 2003) then extended the fraction of unsatisfied demand backordered to any decreasing function of the waiting time up to the next replenishment. Teng and Yang (2004) further generalized the partial backlogging EOQ model to allow for time-varying purchase cost. Yang (2005) made a comparison among various partial backlogging inventory lot-size models for deteriorating items on the basis of maximum profit.

Since, holding cost is an integral part of the total cost of inventory. In fact, holding cost, further, is constituted by the cost of loading and reloading labors, holding facility and energy, rent of warehouse, taxes, insurances, etc. Holding cost depends on the units and time for which units are kept in warehouse. The notion of variable holding cost becomes more important when demands of product is boosted by its level in stock. Therefore, two constraints, level of stocks and conditions of stock, must be discussed in such inventory model. In the scenario of stock dependent demand, high level of stock attracts more demand but at the same time risk of mishandling items creates significant concern of inventory holder. Because, mishandling of items may result in reduction of the profit due to denying of customers to pay full payment for such items. So, every inventory manager should focus on providing good facilities and services for keeping inventory in good conditions.

For that reason, holding cost is highly needed to pay attention in order to control total inventory cost. Weiss (1982) explained that variable holding costs could be appropriate when the value of an item decreases the longer it is in stock; Ferguson et al. (2007) recently stated that this kind of model is suitable for perishable items in which price markdowns or removal of aging product are essential. Goh (1994) first studied a stock-dependent demand model with variable holding costs, where the unit holding cost was a nonlinear continuous function of the time the item is in stock or a nonlinear continuous function of the inventory level. Giri and Chaudhuri (1998) extended this model for perishable products. Alfares (2007) investigated the inventory model with stock-level dependent demand rate and variable holding cost. Mishra and Singh (2011) extended the inventory model for deteriorating items with time needy linear demand and holding cost. To study the concept of variability of the holding cost of decaying item, Tyagi et al. (2012) investigated an inventory model for decaying item with power demand prototype and managed first Weibull function for holding cost rate. In that study, the holding cost depends continuously on deterioration rate and storage epoch, shortages were allowed and partially backlogged inversely with the waiting time for the next replenishment. Tripathi (2013) investigated an inventory model for time varying demand and constant demand; and time dependent holding cost and constant holding cost for case 1 and case 2, respectively. He considered non-decaying items in his model and gave a motivation to study another model for deteriorating items with discrete holding cost. Tyagi et al. (2014) presented an inventory model for deteriorating item with stock-dependent demand and variable holding cost. Furthermore, non-instantaneous deteriorating approach was considered in this work and shortages are allowed and partially backlogged. The optimal policies are derived and the necessary and sufficient conditions of the existence and uniqueness of the optimal solution theoretically are carried out. As the special cases, the results of the proposed model with instantaneous and non-instantaneous deterioration rate and with and without shortages were shown. Tyagi (2014) studied an optimization of inventory model where items deteriorate in stock conditions. In this paper, to generalize the decaying conditions based on the location of warehouse and conditions of storing, the rate of deterioration follows the Weibull distribution function. In addition, demand of fresh item is exponentially declining with time, shortages are allowed and partially backlogged.

Unfortunately, none of the aforesaid papers considered that holding cost depends on the consumption rate of the units from stock. Since, holding cost is related to the rate of change of units from warehouse. Therefore, it is imposed a depletion rate stimulated holding cost when demand rate is a power function of the on-hand inventory down to a certain stock level, at which the demand rate becomes a constant. We also prove that the optimal replenishment policy not only exists but also is unique. Moreover, numerical example is used to illustrate the proposed model, and concluding remarks are provided.

2. Notations and assumptions

2.1. Notations

To develop the mathematical model of inventory replenishment schedule, the notation adopted in this paper is as below:

A	replenishment cost per order,
c	purchasing cost per unit,
s	selling price per unit, where $s > c$,
Q	ordering quantity per cycle,
$I_{\max}$	maximum inventory level per cycle,
$C_1(t)$	holding cost per unit per unit time,
C_2	backorder cost per unit per unit time,
C_3	opportunity cost (i.e., goodwill cost) per unit,
t_1	time at which the inventory level reaches S_0, where S_0 is given,
t_2	time at which the inventory level reaches zero,
t_3	length of period during which shortages are allowed,
T	length of the inventory cycle, hence $T = t_2 + t_3$,
$I_1(t)$	level of positive inventory at time t, where $0 \le t \le t_1$,
$I_2(t)$	level of positive inventory at time t, where $t_1 \le t \le t_2$,
$I_3(t)$	level of negative inventory at time t, where $t_2 \le t \le T$,
$P(t_1, t_3)$	total profit per unit time with two-component demand rate
$P'(t_2, t_3)$	total profit per unit time with constant demand rate.

2.2. Assumptions

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero. Furthermore there is no decay of units due to adopting quick response process for maintaining good environment at manageable cost.

2. The time horizon of the inventory system is infinite.
3. The demand rate is dependent on the on-hand inventory down to a level I_0, where I_0 is given and fixed, beyond which it is assumed to be a constant, that is, when the on-hand inventory level is $I(t)$, the demand rate $R(I(t))$ of the item is considered to be of the form

$$R(I(t)) = \begin{cases} \alpha[I(t)]^{\beta}, & I(t) \geq I_0 \\ D, & 0 \leq I(t) < I_0 \end{cases}$$

where $\alpha > 0$ and $0 < \beta < 1$ are termed as scale and shape parameters respectively, $D(>0)$ is a constant such that $D = \alpha I_0^{\beta}$.
4. Shortages are allowed and the demand rate $R(I(t))$ is given by:

$$R(I(t)) = D, \qquad I(t) < 0:$$

It is adopted the concept used here is that some of the unsatisfied demand is backlogged, and the fraction of shortages backordered is $1/(1+\delta x)$, where x is the waiting time up to the next replenishment and δ is a positive constant.

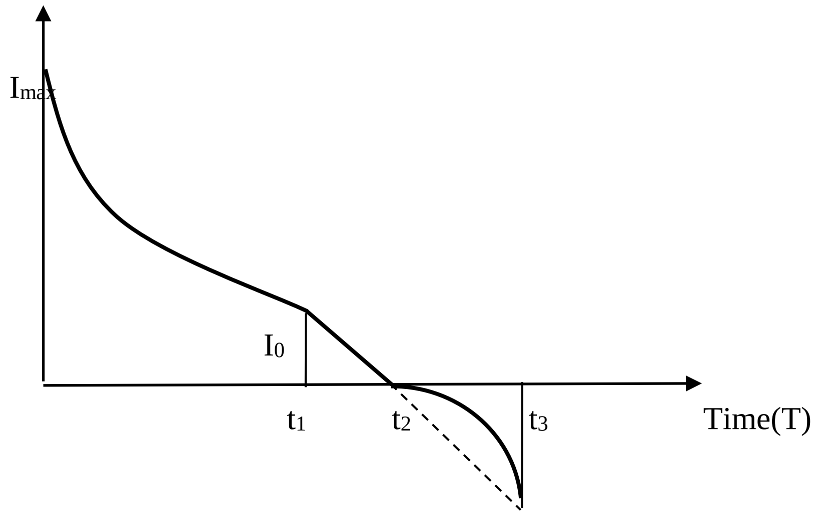

Fig. 1 Graphical representation of the inventory system.

3. Mathematical formulation

In the present model, the parameter I_0 is exogenous. Depending on the constant I_0 and the maximum inventory level $I_{\max}$, the inventory problem here has two situations: (i) $I_{\max} \geq I_0$ and (ii) $I_{\max} < I_0$.

3.1. Inventory problem with $I_{\max} \geq I_0$

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, it is considered the following time intervals separately, $[0, t_1]$, $[t_1, t_2]$ and $[t_2, T]$. During the interval $[0, t_1]$, the inventory is depleted due to the effect of demand dependent on the on-hand inventory level and reaches the level I_0 at time $t = t_1$. Hence, the inventory level is governed by the following differential equation:

$$\frac{dI_1(t)}{dt} = -\alpha\left[I_1(t)\right]^{\beta}, \; 0 < t < t_1$$

with the boundary condition $I_1(t_1) = I_0$. Solving the differential equation, it gets the inventory level as:

$$I_1(t) = \left[I_0^{1-\beta} + \alpha(1-\beta)(t_1 - t) \right]^{\frac{1}{1-\beta}},\ 0 \le t \le t_1$$

After the time $t = t_1$, the demand rate becomes a constant D, and the inventory level falls to zero at time $t = t_2$. During the interval $\left[t_1, t_2\right]$, the inventory is depleted due to the effect of demand. Hence, the inventory level is governed by the following differential equation:

$$\frac{dI_2(t)}{dt} = -D, t_1 \le t \le t_2$$

with the boundary condition $I_2(t_2) = 0$. Solving the differential equation, it obtains the inventory level as

$$I_2(t) = D(t_2 - t),\ \ t_1 \le t \le t_2.$$

Due to the continuity of $I_1(t)$ and $I_2(t)$ at point $t = t_1$, it follows that $I_0 = D(t_2 - t)$, which implies

$$t_2 = t_1 + \frac{I_0}{D}.$$

Thus, t_2 is a function of t_1. Furthermore, at time t_2, shortage occurs and the inventory level starts dropping below zero. During $\left[t_2, T\right]$, the inventory level only depends on demand, and a fraction $\frac{1}{1+\delta(T-t)}$ of the demand is backlogged, where $t \in \left[t_2, T\right]$. The inventory level is governed by the following differential equation:

$$\frac{dI_3(t)}{dt} = -\frac{D}{1+\delta(T-t)},\ t_2 < t < T,$$

with the boundary condition $I_3(t_2) = 0$. Solving the differential equation, model obtain the inventory level as

$$I_3(t) = -\frac{D}{\delta}\left\{In[1+\delta(T-t_2)] - In[1+\delta(T-t)]\right\},\ t_2 \le t \le T.$$ Hence

$$I(t) = \begin{cases} I_1(t) = \left[I_0^{1-\beta} + \alpha(1-\beta)(t_1 - t) \right]^{\frac{1}{1-\beta}}, 0 \le t \le t_1. \\ I_2(t) = D(t_2 - t), \quad t_1 \le t \le t_2. \\ I_3(t) = -\frac{D}{\delta}\left\{In[1+\delta(T-t_2)] - In[1+\delta(T-t)]\right\}, \quad t_2 \le t \le T. \end{cases} \tag{1}$$

Therefore, the ordering quantity over the replenishment cycle can be determined as

$$Q = I_1(0) - I_3(T) = [I_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{1}{1-\beta}} + \frac{DIn(1+\delta t_3)}{\delta}. \tag{2}$$

In addition, the maximum inventory level per cycle is $I = I_1(0) = [I_0^{1-\beta} + \alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}$.

Based on Eq. (1) and Eq. (2), the total profit per cycle consists of the following elements:

1. Ordering cost per cycle $= A$,
2. Actually, holding cost is the composition of fix cost and variable expenditure of maintaining or managing the inventory. In fact, the expenditure to hold a unit increases as the time period for which the unit is held in warehouse increases. Conversely, if a unit does not spend more time in warehouse,

we have not to spend more on it. It is observed that time period of a unit in warehouse or depletion rate decreases holding cost or expenditure to hold a unit decreases. Therefore,

Holding cost per unit per unit time $\propto$ 1/depletion rate of a unit

Using this fact, we have assumed that holding cost per unit per unit time when demand depends upon stock level is

$$C_1(t)=\begin{cases} a+\dfrac{b}{I_1'}, & t\in[0,t_1] \\ a, & t\in[t_1,t_2]. \end{cases}$$

Where, a and b are positive constant with $aD>b$ and b governs the expenditure on managing the item in good conditions. I_1' i.e. $\dfrac{dI_1}{dt}$ rate of change of inventory that is depletion rate during the interval $\left[0,t_1\right]$. Therefore, holding cost per cycle

$$=\frac{a}{\alpha(2-\beta)}\left\{I_0^{2-\beta}-[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{2-\beta}{1-\beta}}\right\}+\frac{aI_0^2}{2D}-\frac{bt_1I_0^{1-\beta}}{\alpha}-\frac{b(1-\beta)t_1^2}{2},$$

3. Backorder cost per cycle $=\dfrac{C_2D}{\delta^2}[\delta t_3-In(1+\delta t_3)]$,

4. Opportunity cost due to lost sales per cycle $=\dfrac{C_3D}{\delta}[\delta t_3-In(1+\delta t_3)]$,

5. Purchase cost per cycle $=cQ=c[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}+\dfrac{cD}{\delta}In(1+\delta t_3)$,

6. Sales revenue per cycle $=sQ=s[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}+\dfrac{sD}{\delta}In(1+\delta t_3)$.

Therefore, the total profit per unit time of the model is obtained as follows:

$$P(t_1,t_3)=\frac{1}{(t_2+t_3)}\Bigg[(s-c)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}-A+\frac{a}{\alpha(\beta-2)}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{2-\beta}{1-\beta}} \quad (3)$$
$$-aI_0^{2-\beta}\left(\frac{1}{\alpha(\beta-2)}+\frac{I_0^{\beta}}{2D}\right)+\frac{b}{\alpha}I_0^{1-\beta}t_1+\frac{b}{2}(1-\beta)t_1^2+D(s-c)t_3$$
$$-\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}[\delta t_3-\ln(1+\delta t_3)]\Bigg].$$

To maximize the total profit per unit time, taking the first partial derivative of $P(t_1,t_3)$ with respect to t_1 and t_3, respectively, we obtain

$$\frac{\partial P(t_1,t_3)}{\partial t_1}=\frac{-1}{(t_1+t_3+\frac{I_0}{D})}\Bigg[P(t_1,t_3)-[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}(\alpha(s-c)$$
$$-a[I_0^{1-\beta}+\alpha(1-\beta)t_1]+\frac{b}{\alpha}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}})\Bigg], \quad (4)$$

and

$$\frac{\partial P(t_1,t_3)}{\partial t_3}=-\frac{1}{t_1+t_3+\frac{I_0}{D}}\left[P(t_1,t_3)-D(s-c)+\frac{D[C_2+\delta(s-c+C_3)]t_3}{1+\delta t_3}\right]. \tag{5}$$

The optimal solution of $P(t_1,t_3)$ must satisfy the equations $\frac{\partial P(t_1,t_3)}{\partial t_1}=0$ and $\frac{\partial P(t_1,t_3)}{\partial t_3}=0$, simultaneously, which implies

$$P(t_1,t_3)=a\alpha(1-\beta)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}\left[\frac{(s-c)}{a(1-\beta)}-\frac{I_0^{1-\beta}}{\alpha(1-\beta)}+\frac{b}{a(1-\beta)\alpha^2}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}}-t_1\right], \tag{6}$$

and

$$P(t_1,t_3)=D(s-c)+\frac{D[C_2+\delta(s-c+C_3)]t_3}{1+\delta t_3}, \tag{7}$$

respectively. Because both the left hand sides in Eq. (6) and Eq. (7) are the same, hence the right hand sides in these equations are equal, that is,

$$\frac{D[C_2+\delta(s-c+C_3)]t_3}{1+\delta t_3}=D(s-c)-a\alpha(1-\beta)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}\left[\frac{(s-c)}{a(1-\beta)}-\frac{I_0^{1-\beta}}{\alpha(1-\beta)}+\frac{b}{a(1-\beta)\alpha^2}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}}-t_1\right]. \tag{8}$$

On the other hand, substituting $P(t_1,t_3)$ in Eq. (3) into Eq. (7) and obtains

$$\begin{aligned}\frac{D[C_2+\delta(s-c+C_3)](t_1+t_3+\frac{I_0}{D})t_3}{1+\delta t_3}&=A-(s-c)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}\\&-\frac{a}{\alpha(\beta-2)}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{2-\beta}{1-\beta}}+aI_0^{2-\beta}\left(\frac{1}{\alpha(\beta-2)}+\frac{I_0^{\beta}}{2D}\right)-\frac{b}{\alpha}I_0^{1-\beta}t_1-\frac{b}{2}(1-\beta)t_1^2\\&+\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}[\delta t_3-\ln(1+\delta t_3)]+D(s-c)(t_1+\frac{I_0}{D}).\end{aligned} \tag{9}$$

Now, here, I want to find the value of (t_1,t_3) which satisfies Eqs. (8) and (9), simultaneously. For convenience, first let $f(t_1)$ denote the right hand side of Eq. (8), that is,

$$f(t_1)=D(s-c)-a\alpha(1-\beta)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}\left[\frac{(s-c)}{a(1-\beta)}-\frac{I_0^{1-\beta}}{\alpha(1-\beta)}+\frac{b}{a(1-\beta)\alpha^2}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}}-t_1\right],\qquad t_1\geq 0\ . \tag{10}$$

It notes that $f(t_1)$ is a continuous function in $t_1 \in [0,\infty)$. Then Eq. (8) becomes

$$f(t_1)=\frac{D[C_2+\delta(s-c+C_3)]t_3}{1+\delta t_3}, \tag{11}$$

which implies,

$$t_3=\frac{f(t_1)}{D[C_2+\delta(s-c+C_3)]-\delta f(t_1)}. \tag{12}$$

Thus, t_3 is a function of t_1, and further we have

$$\frac{dt_3}{dt_1}=\frac{D[C_2+\delta(s-c+C_3)]}{\left\{D[C_2+\delta(s-c+C_3)]-\delta f(t_1)\right\}^2}\frac{df(t_1)}{dt_1}. \tag{13}$$

Furthermore, motivated by Eq. (9), it gets

$$\begin{aligned} g(t_1)&=A-(s-c)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1}{1-\beta}}-\frac{a}{\alpha(\beta-2)}[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{2-\beta}{1-\beta}}+\\ &\quad +aI_0^{2-\beta}\left(\frac{1}{\alpha(\beta-2)}+\frac{I_0^{\beta}}{2D}\right)-\frac{b}{\alpha}I_0^{1-\beta}t_1-\frac{b}{2}(1-\beta)t_1^2+D(s-c)(t_1+\frac{I_0}{D})\\ &\quad +\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}[\delta t_3-\ln(1+\delta t_3)]-\frac{D[C_2+\delta(s-c+C_3)](t_1+t_3+\frac{I_0}{D})t_3}{1+\delta t_3} \end{aligned} \tag{14}$$

where t_3 is given as in Eq. (12). we have to take the derivative of $g(t_1)$ with respect to t_1 and by using the relations shown in Eq. (8), Eq. (10) and Eq. (13), we obtain

$$\begin{aligned} \frac{dg(t_1)}{dt_1}&=-\frac{D[C_2+\delta(s-c+C_3)](t_1+t_3+\frac{I_0}{D})}{(1+\delta t_3)^2}\frac{dt_3}{dt_1}\\ &=-(t_1+t_3+\frac{I_0}{D})\frac{df(t_1)}{dt_1}. \end{aligned} \tag{15}$$

In order to prove the existence and uniqueness of the optimal solution t_1^* which satisfies equation $g(t_1^*)=0$, I have to investigate the property of function $f(t_1)$. Taking the derivative of $f(t_1)$ with respect to t_1, I have

$$\begin{aligned} \frac{df(t_1)}{dt_1}&=-\beta[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{2\beta-1}{1-\beta}}\Bigg[\alpha(s-c)+[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{1-2\beta}{1-\beta}}\\ &\quad \left[b-a\alpha[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}}\right]\Bigg]-b(1-2\beta)+a\alpha(1-\beta)[I_0^{1-\beta}+\alpha(1-\beta)t_1]^{\frac{\beta}{1-\beta}} \end{aligned} \tag{16}$$

Because $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}>0$, since $(aD>b)$ and it can be shown that $lim_{t_1\to\infty} f(t_1)=\infty$, thus, I have the following result.

Lemma 1: Let $f(t_1)$ be defined as in Eq. (10). For given $aD>b$, I have $f(t_1)$ is a strictly increasing function in $t_1\in[0,\infty)$, and the minimum value of $f(t_1)$ is $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}>0$.

Proof: See **Appendix A.**

Now, let us consider the following two sub cases: (i) $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}\ge\frac{D[C_2+\delta(s-c+C_3)]}{\delta}$

(ii) $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}<\frac{D[C_2+\delta(s-c+C_3)]}{\delta}$.

For convenience, we let

$$F(t_3)=\frac{D[C_2+\delta(s-c+C_3)]t_3}{1+\delta t_3}, t_3\ge 0 \tag{17}$$

3.1.1 Case1: When $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}\ge\frac{D[C_2+\delta(s-c+C_3)]}{\delta}$, Eq. (17), becomes

$F(t_3)<\frac{D[C_2+\delta(s-c+C_3)]}{\delta}\le aI_0-\frac{bI_0^{1-\beta}}{\alpha}=f(0)$, for $t_3\in[0,\infty)$. By lemma, therefore $f(t_1)$ is strictly non-decreasing function in $t_1\in[0,\infty)$. Hence for any given $t_1\in[0,\infty)$, there is no value $t_3\in[0,\infty)$ such that $f(t_1)=F(t_3)$, i.e. for any given $t_1\in[0,\infty)$, we cannot find a value t_3 which satisfies Eq. (8). However, for this situation, from Eq. (4), Eq. (7), Eq. (9) and Eq. (17), this have

$\frac{\partial P(t_1,t_3)}{\partial t_1}=\frac{F(t_3)-f(t_1)}{(t_1+t_3+\frac{I_0}{D})}<\frac{F(t_3)-f(0)}{(t_1+t_3+\frac{I_0}{D})}<0$, for any $t_1\in(0,\infty)$ and $t_3\in(0,\infty)$.

Therefore, if $f(0)=aI_0-\frac{bI_0^{1-\beta}}{\alpha}\ge\frac{D[C_2+\delta(s-c+C_3)]}{\delta}$, the maximum value of $P(t_1,t_3)$ occurs at the boundary point $t_1^*=0$. In special circumstance that $t_1^*=0$, the optimal values of t_2 (denoted by t_2^*) can be obtained by $t_2=t_1+\frac{I_0}{D}$ and is $t_2^*=\frac{I_0}{D}$. Then the total profit per unit time in Eq. (3) becomes,

$$P(t_3)\equiv P(0,t_3)=D(s-c)-\frac{1}{(t_3+\frac{I_0}{D})}\left[A+\frac{aI_0^2}{2D}+\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}[\delta t_3-\ln(1+\delta t_3)]\right] \tag{18}$$

The necessary condition to find the optimal solution of $P(t_3)$ is $\frac{dP(t_3)}{dt_3}=0$, which implies

$$A+\frac{aI_0^2}{2D}+\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}[\delta t_3-\ln(1+\delta t_3)]-\frac{D[C_2+\delta(s-c+C_3)]t_3(t_3+\frac{I_0}{D})}{1+\delta t_3}=0. \tag{19}$$

Let $Z(t_3) = A + \frac{aI_0^2}{2D} + \frac{D[C_2 + \delta(s - c + C_3)]}{\delta^2}[\delta t_3 - \ln(1 + \delta t_3)] - \frac{D[C_2 + \delta(s - c + C_3)]t_3(t_3 + \frac{I_0}{D})}{1 + \delta t_3}$. The derivative of $Z(t_3)$ with respect to t_3 is

$$\frac{dZ(t_3)}{dt_3} = -\frac{D[C_2 + \delta(s - c + C_3)](t_3 + \frac{I_0}{D})}{(1 + \delta t_3)^2} < 0,$$

thus, $Z(t_3)$ is strictly decreasing function in $t_3 \in [0, \infty)$. Furthermore we have $Z(0) = A + \frac{aI_0^2}{2D} > 0$ and $\lim_{t_3 \to \infty} Z(t_3) < 0$. By using the intermediate value theorem, there exists a unique solution $t_3^* \in (0, \infty)$ such that $Z(t_3^*) = 0$, that is, t_3^* is a unique value which satisfies Eq. (19). Summarize the above arguments; it is obtained the following theorem

Theorem 1: For $aI_0 - \frac{bI_0^{1-\beta}}{\alpha} \geq \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}$, the optimal value of (t_1, t_2, t_3) is given by $t_1^* = 0, t_2^* = \frac{I_0}{D}$ and t_3^* is the value which satisfies Eq. (19).

When, $t_1^* = 0$ the inventory problem becomes the regular EOQ with constant demand rate and partial backordering. We have to obtain once the optimal value $(t_1^*, t_3^*) = (0, t_3^*)$ is obtained by Eq. (18) and Eq. (19), the maximum total profit per unit time is as follows.

$$P(t_3^*) = D(s - c) + \frac{D[C_2 + \delta(s - c + C_3)]t_3^*}{1 + \delta t_3^*} \tag{20}$$

The maximum inventory level per cycle is $I^* = I_0$.

3.1.2. Case 2: Where $f(0) = aI_0 - \frac{bI_0^{1-\beta}}{\alpha} < \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}$, from lemma 1, $f(t_1)$ is a strictly increasing function of $t_1 \in [0, \infty)$, thus a unique value of $\hat{t}_1 \in (0, \infty)$ can be found such that $f(\hat{t}_1) = \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}$. Furthermore, for any given $t_1 \geq \hat{t}_1$, I have

$$f(t_1) \geq f(\hat{t}_1)$$
$$= \frac{D[C_2 + \delta(s - c + C_3)]}{\delta} > \frac{D[C_2 + \delta(s - c + C_3)]}{\delta}\left[1 - \frac{1}{1 + \delta t_3}\right] = F(t_3).$$

It implies that a value $t_3 \in [0, \infty)$ cannot be obtained such that Eq. (8) holds. Therefore, the optimal solution of t_1 which satisfies Eq. (8) will occur in the interval $(0, \hat{t}_1)$. On the other hand, from the definition of $F(t_3)$ in Eq. (17), it can be shown that $F(t_3)$ is a continuous and strictly increasing function in $t_3 \in [0, \infty)$.

Besides, we have $F(0)=0$, and $\lim_{t_3\to\infty} F(t_3)=\dfrac{D[C_2+\delta(s-c+C_3)]}{\delta}=f(\hat{t}_1)>f(t_1)$, for any $t_1\in[0,\hat{t}_1)$. Thus, for any given $t_1\in[0,\hat{t}_1)$, there exists a unique value $t_3\in(0,\infty)$ such that $F(t_3)=f(t_1)$.

Consequently, for any given to $t_1\in[0,\hat{t}_1)$, when $f(0)=aI_0-\dfrac{bI_0^{1-\beta}}{\alpha}<\dfrac{D[C_2+\delta(s-c+C_3)]}{\delta}$, a unique value $t_3\in(0,\infty)$ can be come up satisfying Eq. (8). Therefore, optimal value $t_1^*\in[0,\hat{t}_1)$ is obtained, the optimal solutions of t_2, t_3 and T are as follows,

$$t_2^*=t_1^*+\frac{I_0}{D} \tag{21}$$

$$t_3^*=\frac{f(t_1^*)}{D[C_2+\delta(s-c+C_3)]-\delta f(t_1^*)} \tag{22}$$

$$T^*=t_2^*+t_3^* \tag{23}$$

Now, it is required to prove the existence and uniqueness of t_1^* in $(0,\hat{t}_1)$. By using $(t_1+t_3+\dfrac{I_0}{D})>0$ and $\dfrac{df(t_1)}{dt_1}>0$ for $t_1\in(0,\hat{t}_1)$ and from Eq. (15), we obtain $\dfrac{dg(t_1)}{dt_1}<0$. Therefore, $g(t_1)$ is a strictly decreasing function in $t_1\in[0,\hat{t}_1)$. Furthermore, from Eq. (12), we have $t_3\to\infty$ as $t_1\to\hat{t}_1^-$, and $\lim_{t_1\to\hat{t}_1^-} g(t_1)=-\infty<0$ and

$$g(0)=A+\left(aI_0-\frac{bI_0^{1-\beta}}{\alpha}\right)\left(\frac{1}{\delta}-\frac{I_0}{2D}\right)+\frac{bI_0^{2-\beta}}{2D\alpha}$$
$$-\frac{D[C_2+\delta(s-c+C_3)]}{\delta^2}In\left\{\frac{D[C_2+\delta(s-c+C_3)]}{D[C_2+\delta(s-c+C_3)]-\delta\left(aI_0-\dfrac{bI_0^{1-\beta}}{\alpha}\right)}\right\} \tag{24}$$

Note that the value in the brace is well defined, becomes as we have $\dfrac{D[C_2+\delta(s-c+C_3)]}{\delta}>f(0)=\left(aI_0-\dfrac{bI_0^{1-\beta}}{\alpha}\right)$. Then, we have the following result.

Lemma 2: For $\dfrac{D[C_2+\delta(s-c+C_3)]}{\delta}>\left(aI_0-\dfrac{bI_0^{1-\beta}}{\alpha}\right)$ we have;

(a) If $g(0)\le 0$, then the optimal value of t_1 is $t_1^*=0$.

(b) If $g(0)>0$, then the solution $t_1^*\in(0,\hat{t}_1)$ which satisfies Eq. (9) not only exist but also is unique.

Proof: See **Appendix B.**

Lemma 2(a) indicates that if $\frac{D[C_2+\delta(s-c+C_3)]}{\delta} > \left(aI_0 - \frac{bI_0^{1-\beta}}{\alpha}\right)$ and $G(0) \le 0$ then the optimal time at which the inventory level reaches I_0 is $t_1^* = 0$. It implies the maximum inventory level in this system is $I^* = I_0$. The corresponding value of t_3^* can be found from Eq. (22) and is given by $t_3^* = \frac{f(0)}{D[C_2+\delta(s-c+C_3)]-\delta f(0)}$. However, since $P\left(0, \frac{f(0)}{D[C_2+\delta(s-c+C_3)]-\delta f(0)}\right) \le$ $P(0,t_3^*) = P(t_3^*)$, where t_3^* is the value of t_3 which satisfies Eq. (19). Thus, t_3^* is the optimal value of t_3 and the maximum total profit per unit time $P(t_3^*)$ is obtained in Eq. (20).

Lemma 2(b) reveals that if $\frac{D[C_2+\delta(s-c+C_3)]}{\delta} > \left(aI_0 - \frac{bI_0^{1-\beta}}{\alpha}\right)$ and $g(0) > 0$, then $t_1^* \in (0, \hat{t}_1)$ and is unique. Furthermore, the unique solution will be proved to be indeed a global maximum point by checking the second order optimality conditions, that is, I have the following main result.

Theorem 2: For $\frac{D[C_2+\delta(s-c+C_3)]}{\delta} > \left(aI_0 - \frac{bI_0^{1-\beta}}{\alpha}\right)$, if $g(0) > 0$, then the point (t_1^*, t_3^*) which satisfies Eqs. (8) and (9) simultaneously is the global maximum point of the total profit per unit time.

Proof: See **Appendix C.**

Once the optimal solution (t_1^*, t_3^*) is obtained, (t_1^*, t_3^*) is substituted into Eq. (3), optimal ordering quantity per cycle, Q^*, and the maximum total profit per unit time $P(t_1^*, t_3^*)$, are as follows,

$$
\begin{aligned}
&Q^* = [I_0^{1-\beta} + \alpha(1-\beta)t_1^*]^{\frac{1}{1-\beta}} + \frac{DIn(1+\delta t_3^*)}{\delta} \text{ and} \\
&P(t_1^*, t_3^*) = D(s-c) + \frac{D[C_2+\delta(s-c+C_3)]t_3^*}{1+\delta t_3^*}
\end{aligned}
\tag{25}
$$

3.2. Inventory problem with $I_{\max} < I_0$

When the stock level I_0 at which the demand rate amends from being inventory level dependent to a constant D is relatively lofty, an optimal inventory manage policy would never order enough to climb $I_{\max}$ to I_0. Under this situation, that $I_{\max} < I_0$, the demand rate is never a function of the inventory level and is always the constant D. That is, what is being discussed in this section is the same as what was discussed in the previous section, but with a very bulky value of I_0. Moreover, the inventory setup becomes the regular EOQ with constant demand rate and holding cost per unit per unit time with partial backordering.

4. Numerical examples

To illustrate the above results, the proposed analytic solution procedure is applied efficiently to solve the following numerical example.

Example: Consider an inventory system with the following characteristics: $s = 15, c = 6, a = 0.8, C_2 = 0.4, A = 100, \alpha = 5, \beta = 0.4$ and $b = 0.024, I_0 = 80, C_3 = 0.7, \delta = 1$ in appropriate units. Applying the proposed way of solution yields the optimal solution with $t_1^* = 5.216992, P^* = 204.412, Q^* = 287.935$

For the given values of different parameters and different values of b and maximizing the objective function for proposed model, I obtain the results which are shown in the Table 1**.**

Table 1
The results of the implementation of the proposed method

Value of 'b'	time t_1^*	Maximum Total profit per unit time	Optimal ordering quantity Q^*
0.026	5.217388	204.421	287.953
0.025	5.217191	204.416	287.944
0.024	5.216992	204.412	287.935
0.023	5.216795	204.408	287.926
0.022	5.216597	204.404	287.917

It can be seen from the above table that as the value of b increases, the value of decision variables increases and vice-versa. On going to find out its reason, it is assumed in this proposed model that holding cost reversely depends upon depletion rate and governed by parameter b . b indicates the level of expenditure on services and facilities provided to keep inventory in consumer friendly conditions. Therefore, depletion rate decreases as the value of b increases and holding cost goes up in same manner but time t_1^* and maximum total profit per unit time P^* increase due to having positive impact on demand side by the high level of stock. Furthermore, this situation allows to maintaining a high level of optimal ordering quantity Q^* . Therefore, this study gives a very interesting managerial insight that whenever demand depends upon stock in hand then increasing the level of expenditure on handling place has positive impact on profit maximizing.

5 Conclusions

Stock dependent inventory models are normally developed with a constant holding cost. But in general holding cost is not always constant. More practically, it can be estimated by depletion rate when demand depends on stock in hand. Therefore, in this paper, for the first time inventory model with stock dependent demand has been considered with depletion rate stimulated holding cost. The proposed study shows that better services and facilities pays better profit. The proposed model can be extended in several ways. For instance, we may consider the permissible delay in payments. Also, we could extend the deterministic demand function to stochastic fluctuating demand patterns. Finally, we could generalize the model to allow for quantity discounts, inflation and others.

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Appendix A. The proof of Lemma 1

Given $aD > b$ .So, the first term in the equation (16) will be positive. From Eqs (16), we know that $\frac{df(t_1)}{dt_1} > 0$. Hence $f(t_1)$ is a strictly increasing function in $t \in [0, \infty)$ and $f(0) = aI_0 - \frac{bI_0^{1-\beta}}{\alpha} > 0$ is its minimum.

Appendix B. The proof of Lemma 2

(a) First, we consider $g(0) < 0$. Since $g(t_1)$ is strictly decreasing in the interval $\left[0, \hat{t}_1\right)$, we cannot find a value $t_1 \in \left[0, t_1\right)$ such that $g(t_1) = 0$. However, by Eqs. (10), (11) and (14), Eq. (4) becomes

$\frac{\partial P(t_1,t_3)}{\partial t_1}=\frac{g(t_1)}{(t_1+t_3+\frac{I_0}{D})^2}$. For this situation that $I(0)<0$, we have $\frac{\partial P(t_1,t_3)}{\partial t_1}<0$ and any $t_1\in[0,\hat{t}_1)$ which implies that for any fixed $t_3\in[0,\infty)$, a smaller value of t_1 cause a higher value of $P(t_1,t_3)$. Therefore the maximum value of $P(t_1,t_3)$ occurs at the boundary point $t_1^*=0$ .Next, if $g(0)=0$, then from the property that $g(t_1)$ is strictly decreasing in the interval $[0,\hat{t}_1)$. We see $t_1^*=0$ is the unique value which satisfies $g(t_1^*)=0$.

(b) If $g(0)>0$, since $g(t_1)$ is strictly decreasing function in $t_1\in[0,\hat{t}_1)$, and $\lim_{t_1\to\hat{t}_1^-} g(t_1)=-\infty<0$, by using the intermediate value theorem, there exist a unique value solution $t_1^*\in(0,\hat{t}_1)$ such that $g(t_1^*)=0$, that is, t_1^* is the unique solution which satisfies Eq. (9). This completes the proof.

Appendix C. The proof of Theorem 2

From lemma 2(b), the solution $t_1^*\in(0,\hat{t}_1)$ which satisfies Eq. (9) not only exists but also is unique. Hence, the value t_3^* can be uniquely determined by Eq. (12). Furthermore, we can obtain

$$\left.\frac{\partial^2 P(t_1,t_3)}{\partial t_1^2}\right|_{(t_1,t_3)=(t_1^*,t_3^*)}=-\frac{1}{(t_1^*+t_3^*+\frac{I_0}{D})}\left.\frac{df(t_1)}{dt_1}\right|_{(t_1,t_3)=(t_1^*,t_3^*)}<0,$$

$$\left.\frac{\partial^2 P(t_1,t_3)}{\partial t_3^2}\right|_{(t_1,t_3)=(t_1^*,t_3^*)}=-\frac{D[C_2+\delta(s-c+C_3)]}{(t_1^*+t_3^*+\frac{I_0}{D})(1+\delta t_3^*)^2}<0,$$

and $\left.\frac{\partial^2 P(t_1,t_3)}{\partial t_1\partial t_3}\right|_{(t_1,t_3)=(t_1^*,t_3^*)}=0$.

Thus, the determinant of the Hessian matrix at the stationary point (t_1^*,t_3^*) is

$$|H|=\frac{D[C_2+\delta(s-c+C_3)]}{(t_1^*+t_3^*+\frac{I_0}{D})^2(1+\delta t_3^*)^2}\left.\frac{df(t_1)}{dt_1}\right|_{(t_1,t_3)=(t_1^*,t_3^*)}>0. \tag{26}$$

Consequently, we can conclude that the stationary point (t_1^*,t_3^*) for our optimization problem is a global maximum. This completes the proof.