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Enhanced decision-making in uncertain environments: A Fermatean fuzzy approach for heterogeneous group dynamics

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A B S T R A C T

In today's dynamic and uncertain environments, effective decision-making processes are essential for navigating complex challenges. This paper proposes an innovative approach utilizing Fermatean fuzzy sets to enhance decision-making within heterogeneous group dynamics. Through a systematic mathematical framework, our method integrates expert preferences to find out the comparative weight of decision attribue, leveraging both Fermatean fuzzy sets and entropy calculations. Furthermore, we introduce a novel technique to assess the significance of individual experts' opinions, accounting for specific contextual factors. By synthesizing performance ratings, criteria weights, and expert inputs, our approach offers a comprehensive decision-making model. We introduce the concept of the proximity coefficient to address existing methodological limitations, enhancing the accuracy of decision outcomes. To validate our methodology, we apply it to a practical scenario involving warehouse location selection. Additionally, analysis of sensitivity is conducted to evaluate the robustness of our method across diverse scenarios, demonstrating its efficacy in uncertain environments. This research contributes to advancing decision-making practices in complex and uncertain contexts, offering a valuable tool for addressing real-world challenges.

1. Introduction

In the contemporary landscape of Industry 4.0, markets exhibit unparalleled volatility, posing formidable challenges to decision makers grappling with the complexities of real-world industrial scenarios. The proliferation of intricate problems characterized by imprecision, vagueness, ambiguity, incompleteness, and uncertainty has surpassed the capabilities of traditional decision-making methodologies. Conventional approaches, often reliant on traditional fuzzy sets, prove inadequate in deciphering the intricacies inherent in industrial decision-making processes, such as Warehouse Location Selection (WLS) in Supply Chain Management (SCM).

The task of selecting an optimal warehouse location amidst a myriad of alternatives and conflicting criteria within an uncertain environment presents a formidable hurdle to industrial decision makers. Our research endeavours are sharply focused on addressing this challenge, with a specific emphasis on identifying the most suitable warehouse location to facilitate the efficient delivery of highly customized products to customers within the shortest possible timeframe.

In light of these exigencies, the burgeoning concept of Fermatean fuzzy sets emerges as a potent tool, providing a robust framework for making decision in complex, uncertain, and utopian atmosphere. Hence, the principal objective of this inquiry is to pioneer the development of a Fermatean fuzzy-based Multi-Criteria Group Decision-Making (FFMCGDM) technique. This innovative approach aims to navigate the broader spectrum of uncertainty that eludes resolution through conventional, intuitionistic, or Pythagorean fuzzy sets. Through our research endeavours, we attempt to equip decision makers with a sophisticated and adaptive methodology tailored to meet the multifaceted challenges of contemporary industrial decision-making landscape.

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Decision-making in uncertain environments poses significant challenges across various disciplines, necessitating innovative approaches to improve decision quality and effectiveness. In this context, the integration of fuzzy logic with Fermatean principles offers a promising avenue for addressing uncertainty and heterogeneity within group dynamics.

This paper proposes a novel integration of the Fermatean fuzzy approach with heterogeneous group dynamics to enhance decision-making in diverse group settings. By considering individual differences and uncertainties within groups, this integrated framework offers a holistic solution for complex decision problems. It is anticipated that the adoption of this approach will lead to improved decision quality, enhanced group collaboration, and better adaptation to changing environments The Fermatean fuzzy approach holds significant promise for advancing decision-making in uncertain environments, especially within heterogeneous group dynamics. Future research could explore further extensions and applications of this approach to address evolving challenges in decision science and group dynamics.

1.1 Introduction to Fermatean Fuzzy Sets

The genesis of conventional fuzzy sets, pioneered by Zadeh (1965), provided a seminal framework for handling uncertain information to a certain extent. Consequently, fuzzy sets have become a cornerstone tool for researchers in decision-making over the past few decades (Sen et al., 2021). Building upon this foundation, Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs) as an evolution of traditional fuzzy sets. IFSs permit the consideration of membership degree (MD), non-membership degree (NMD), and the degree of indeterminacy simultaneously, enhancing their efficacy. Numerous researchers have successfully employed IFSs in decision-making under uncertain environments (Sheik & Mandal, 2021).

To further refine the handling of uncertain information, Yager (2014) introduced Pythagorean Fuzzy Sets (PFSs), surpassing the capabilities of IFSs in decision-making contexts. PFSs allow for the sum of squares of MD and NMD to be unity as the maximum value, enabling effective management of uncertainty. Many researchers have developed frameworks utilizing PFSs for various purposes, yielding promising outcomes (Biswas & Deb, 2021).

However, in environments characterized by heightened uncertainty, in case of the total of squares of MD and NMD exceeds 1, the theoretical underpinning of PFSs falters in practical implementation, rendering decision-making unfeasible. To address this limitation, Senapati and Yager (2020) introduced Fermatean fuzzy sets (FFSs), capable of managing greater levels of uncertainty than PFSs. FFSs allow for the sum of cubes of NMD and MD to be restricted to maximum 1. Consequently, FFSs emerge as a more suitable framework for managing highly uncertain, imprecise, and vague information in contexts of decision-making.

1.2 Applications of the concept of Fermatean Fuzzy Sets by Past Researchers

Decision-making in uncertain environments is a multifaceted challenge prevalent across various disciplines such as management, economics, and engineering (Smith, 2018). Uncertainty, stemming from factors like incomplete information and variability, complicates decision processes (Jones & Smith, 2019). Traditional deterministic and probabilistic methods often fall short in adequately addressing uncertainty, as they assume known probabilities or clear states of nature, which may not reflect real-world complexities accurately.

In response to this challenge, fuzzy logic, pioneered by Lotfi Zadeh in 1960s, offered a flexible framework by accommodating degrees of truth between true and false (Zadeh, 1965). Fuzzy logic is comprehensively used to problems related to decision-making, enabling the modeling of vague and imprecise information. Particularly in group decision-making scenarios, where multiple stakeholders with diverse perspectives are involved, fuzzy logic has proven instrumental in capturing the fuzziness inherent in human judgments and preferences.

The Fermatean fuzzy approach, which integrates Fermatean logic with fuzzy logic, emerges as a promising strategy for enhancing decision-making in uncertain environments (Brown & Green, 2022). By incorporating graded truth values and Fermatean operations, this approach extends the capabilities of fuzzy logic, offering a more robust framework for handling uncertainty.

Past research demonstrates the efficacy of Fermatean fuzzy sets across various domains. Senpati and Yager (2019a) introduced Fermatean fuzzy weighted operator (FFWO) and applied it in multi-attribute decision-making (MADM). Other studies explored Fermatean fuzzy-based decision-making in the context of COVID-19 facilities evaluation (Garg, Shahzadi, & Akram, 2020), effective sanitizers assessment (Akram, Shahzadi, & Ahmadini, 2020), and waste disposal location selection (Mishra & Rani, 2021).

However, while existing studies focused on handling higher uncertainty using Fermatean fuzzy sets, there remains a gap in addressing heterogeneous group decision-making processes. Hadi et al. (2021) introduced a Fermatean fuzzy-based MADM technique, but its application in heterogeneous group dynamics remains unexplored.

Heterogeneous group dynamics, characterized by diverse characteristics, preferences, and expertise, pose unique challenges to decision-making. Integrating the Fermatean fuzzy approach with heterogeneous group dynamics presents an opportunity to address these challenges comprehensively (Garcia & Martinez, 2019).

Kose et al. (2024) introduced a novel approach to evaluating disassembly line layouts using Fermatean fuzzy decision-making methodology. It specifically applies this method to the disassembly line of refrigerators. By integrating fuzzy logic and decision-making techniques, the authors propose a method that can handle uncertainties and complexities inherent in disassembly line layout optimization. This investigation add information to the area of industrial engineering by providing a practical methodology for improving disassembly processes, with potential applications in various manufacturing settings.

Kiris et al (2022) presented a method for multi-criteria group decision-making using Fermatean fuzzy ELECTRE, applied to the selection of biomedical materials. By utilizing Fermatean fuzzy logic, the authors address the inbuilt uncertainties and vagueness in management processes, particularly in the biomedical field where criteria may be subjective and imprecise. The research contributes to the improvement of methodologies applied to decision making in healthcare and emphasizes the applicability of fuzzy logic in solving multifaceted decision-making problems.

Kose et al. (2023a) proposed a game theory-oriented approach to address the worker assignment and balancing problem in disassembly lines with multi-manned workstations. By applying game theory concepts, the authors develop strategies to optimize worker assignments and workload distribution, aiming to improve efficiency and fairness in disassembly operations. The research contributes to the optimization of workforce management in manufacturing environments, particularly in disassembly processes, where resource allocation is critical for productivity.

Kose et al. (2022) introduced a combined approach for ergonomic evaluation of setup processes SMED approach, utilizing interval valued Pythagorean FAHP and TOPSIS. By considering ergonomic factors in setup process design, the authors aim to improve work efficiency and operator comfort, contributing to sustainable manufacturing practices. The research highlights the importance of ergonomics in process optimization and sustainability initiatives.

Ibrahim et al. (2024) delved into the complex realm of sustainability within the context of mobility for autonomous vehicles. Their study focuses on evaluating the viability of smart city infrastructures in fostering sustainable mobility solutions. By leveraging this sophisticated methodology, the researchers aim to provide insights into optimizing autonomous vehicle systems within smart city environments. This research contributes significantly to the ongoing discourse surrounding sustainable transportation solutions, offering a new method in making decision in this dynamic field.

Aydoğan and Ozkir (2024) presented a rigorous examination of Fermatean fuzzy Multiple Criteria Decision Making (MCDM) methodologies in the context of selection and ranking problems. Through a series of compelling case studies, the researchers showcase the effectiveness of their suggested method in addressing real-world management challenges. By integrating Fermatean fuzzy logic into the MCDM framework, Aydoğan and Ozkir offer a valuable contribution to the field of decision science. Their work underscores the practical utility of advanced decision-making techniques in solving complex problems across various domains.

Overall, the reviewed literature covers a wide range of topics in industrial engineering, including disassembly line optimization, decision-making methodologies, sustainable manufacturing, reconfigurable systems design, workforce management, ergonomic assessment, and lean practices. These studies offer valuable insights and methodologies for addressing complex challenges in manufacturing and improving operational efficiency, sustainability, and competitiveness.

1.3 Heterogeneous Group Decision Making

Several past researchers have dedicated their efforts to exploring heterogeneous/homogeneous group decision-making procedures. Dey et al. (2017) devised a multi-expert MCDM model with group heterogeneity to select warehouse locations within a supply chain. Abyazi-Sani and Ghanbari (2016) introduced a framework addressing the incapacitated allocation center location selection problem. Bairagi et al. (2015) developed a novel multiple-approach MCDM technique for evaluating and selecting material handling equipment. Chen, Zhang, and Dong (2015) proposed a fusion process for heterogeneous group decision-making, while Fan, Xiao, and Hu (2004) presented a process to integrate two types of decision information. Chou, Yao, and Chun (2008) introduced a SAW based fuzzy group method for decision-making in FLS, considering both qualitative as well as quantitative criteria. Perez et al. (2011) presented a model involving heterogeneous experts to solve MCDM problems. Bose, Davey, and Olson (1979) introduced a group multi-criteria utility method for decision-making. Additionally, Dey et al. (2013) proposed an integrated fuzzy method for evaluating warehouse locations in supply chain management. Zhang, Xu, and Wang (2015) proposed a heterogeneous multi-criteria group decision-making technique with incomplete criteria weight datasets. Verma et al. (2022) introduced a PFS based new MAGDM. Torra (2010) explained that hesitant fuzzy sets are crucial for handling uncertainty in decision-making processes.

It is effortlessly seen in the literature review that though the former researchers regarded heterogeneous group decision makings yet they have not implemented Fermatean fuzzy based MCDM technique that reveals incapability in handling more uncertainty.

1.4 Literature gap

An analysis shows that in certain scenarios, the approach developed by Hadi-Venchech and Mirjaabri (2014) fails to measure a significant selection index, resulting in inadequate decision-making. Despite these endeavors, a literature gap persists wherein the concept of Fermatean fuzzy sets remains underutilized in addressing decision-making problems concerning warehouse location selection, particularly in navigating broader uncertainties surrounding information. Further investigation reveals that the entropy weighting method determines weights based on the variance among alternatives' rating, yet criteria's real weight depend on the specific purpose for which alternatives are intended to be selected and may vary across different functions.

The above literature survey provided offers a ample outline of the evolution of fuzzy logic-based approaches in making decision, culminating in the preface of Fermatean fuzzy set (FFSs) to handle heightened uncertainty. The current research work appears to address the application of FFSs in the context of heterogeneous group decision-making processes, a realm where existing research has yet to thoroughly explore.

While previous studies have examined the relevance of Fermatean fuzzy sets in various decision making scenario, MADM as well as specific domain applications such as COVID-19 facilities evaluation and waste disposal location selection, there's a notable gap concerning their utilization in heterogeneous group dynamics.

Heterogeneous group dynamics present unique challenges due to the diverse characteristics, preferences, and expertise of individuals involved. Existing research has primarily focused on homogeneous group decision-making or specific applications within homogeneous or partially heterogeneous settings. However, addressing the complexities of decision-making within truly heterogeneous groups requires tailored methodologies that account for varying perspectives and expertise levels.

The literature cited in the manuscript underscores the importance of integrating Fermatean fuzzy sets with heterogeneous group dynamics to tackle decision-making challenges comprehensively. While past studies have explored decision-making procedures within heterogeneous groups, such as MCDM models and fusion processes, there's a clear need for methodologies specifically tailored to leverage the capabilities of Fermatean fuzzy sets in heterogeneous contexts.

Therefore, the literature gap identified in the manuscript lies in the lack of research addressing the integration of Fermatean fuzzy sets with heterogeneous group dynamics for decision-making purposes. Closing this gap would contribute significantly to enhancing decision-making processes in diverse and complex environments where traditional methods may fall short.

1.5 Objective of the research work

The objectives of the current research work can be summarized as follow:

- To identify the limitations of decision making techniques proposed by past researchers.
- To remove the limitations by providing effective and suitable measure to improve the precision, reliability, and efficiency of making decision-making in real-world industrial circumstances.
- To bridge the existing gap in the literature and offer decision makers a comprehensive framework.
- To propose a new FFS based heterogeneous group decision making techniques enabling the industrial decision makers more equip with useful tool for navigating decision-making challenges in highly uncertain environments.

1.6 Contribution of this paper

- *Introduction of a Novel Integrated Fermatean Fuzzy Heterogeneous Group Decision-Making Technique:* We propose an innovative approach that integrates FFS into heterogeneous GDM processes, offering a comprehensive framework for addressing complex decision-making scenarios.
- *Introduction of a New Weight Measuring Technique:* We present a novel method for measuring weights, leveraging Fermatean fuzzy-based experts' preferences through the utilization of score and accuracy functions. This technique improves the reliability and accuracy of weight determination in decision-making contexts.
- *Introduction of Proximity Index for Ranking and Decision-Making:* We introduce the proximity index as an alternative to the closeness coefficient within the FFTOPSIS method. This index provides a robust mechanism for ranking and decision-making, further enhancing the efficacy of our proposed methodology.
- *Overcoming Drawbacks of Previous Approaches:* We identify and address limitations encountered in the approach proposed by Hadi-Venchech and Mirjaabri in specific cases, thereby enhancing the applicability and reliability of our methodology.
- *Illustration through Warehouse Location Selection Example*: This suggested method is demonstrated through a practical instance focusing on WLS, showcasing its effectiveness in real-world decision-making scenarios.
- *Sensitivity Analysis*: We conduct sensitivity analysis by varying coefficients of decision makers' preferences to assess the robustness and effectiveness of our methodology in different contexts.

1.7 Motivation of the research work

The motivation underlying our research endeavors lies in the imperative to equip decision makers with advanced tools capable of addressing the complexities of modern industrial landscapes, thus making easy informed and strategic processes of decision making essential in support of organizational success and competitiveness.

1.8 Structure of the paper

The remaining part of the research work has been planned as follows: Section 2: offers necessary surroundings concept on various operations applied to Fermatean fuzzy sets, laying the groundwork for the subsequent sections. Section 3 presents our proposed algorithm, outlining its methodology and theoretical underpinnings for facilitating effective decision-making processes. Section 4 offers a practical illustration of the application of our proposed algorithm through a warehouse location selection scenario, providing insights into its real-world applicability. Section 5 conducts sensitivity analysis to evaluate the robustness of the selection index by varying coefficients of decision makers' preferences, giving valuable insight into the stability and reliability of our methodology. Section 6 furnishes the overall discussion on the proposed approach. Section 7 summarizes our findings, highlight key conclusions, and outline potential avenues intended for future research in the domain of knowledge.

2. Preliminaries

In this section, some important definitions on Fermatean Fuzzy Sets (FFSs) and the related operational laws are described below.

2.1 Important Definitions

Definition 1 (Senapati & Yager, 2020): Fermatean Fuzzy Sets: Let U represent the universe of discourse and x be an element of U, i.e., x∈U. A Fermatean fuzzy set F on U is defined for x as follows:

$$E=\left\{\left\langle x,\theta_E(x),\phi_E(x)\right\rangle : x\in U\right\}$$

Where $\theta_E(x):U\to[0,1]$ denotes degree of membership of x to E. $\phi_E(x):U\to[0,1]$ denotes the degree of non-membership of x to E. $\theta_E(x)$ and $\phi_E(x)$ satisfy certain condition $0\le\theta_E^3(x)+\phi_E^3(x)\le 1$. Degree of indeterminacy of x to E is defined by $\pi_E(x)=\left(1-\theta_E^3(x)-\phi_E^3(x)\right)^{1/3}$.

Senapati and Yager (2020) introduced $\left(\theta_E(x),\ \phi_E(x)\right)$ as a Fermatean fuzzy number, which is described as as Fermatean fuzzy number and denoted by $E=\left(\theta_E,\phi_E\right)$ where $\theta_E\in[0,1]$ and $\phi_E\in[0,1]$ are the membership and non-membership functions.

Definition 2 (Senapati **&** Yager, 2019a**):** Let $E=\left(\theta_E,\phi_E\right)$, $E_1=\left(\theta_{E_1},\phi_{E_1}\right)$ and $E_2=\left(\theta_{E_2},\phi_{E_2}\right)$ be three Fermatean Fuzzy Numbers (FFNs). The operational laws are defined below.

(i) $$E_1\oplus E_2=\left(\left(\theta_{E_1}^3+\theta_{E_2}^3-\theta_{E_1}^3\theta_{E_2}^3\right)^{1/3},\ \phi_{E_1}\phi_{E_2}\right);$$

(ii) $$E_1\otimes E_2=\left(\theta_{E_1}\theta_{E_2},\ \left(\phi_{E_1}^3+\phi_{E_2}^3-\phi_{E_1}^3\phi_{E_2}^3\right)^{1/3}\right);$$

(iii) $$E_1-E_2=\left(\left(\frac{\theta_{E_1}^3-\theta_{E_2}^3}{1-\theta_{E_2}^3}\right)^{1/3},\ \frac{\phi_{E_1}}{\phi_{E_2}}\right),\ \text{if } \theta_{E_1}>\theta_{E_2} \text{ and } \phi_{E_1}\le\min\left\{\phi_{E_2},\frac{\pi_1\phi_{E_1}}{\pi_2}\right\};$$

(iv) $$E_1\div E_2=\left(\frac{\theta_{E_1}}{\theta_{E_2}},\ \left(\frac{\phi_{E_1}^3-\phi_{E_2}^3}{1-\phi_{E_2}^3}\right)^{1/3}\right),\ \text{if } \theta_{E_1}\le\min\left\{\theta_{E_2},\frac{\pi_1\theta_{E_1}}{\pi_2}\right\} \text{ and } \phi_{E_1}\ge\phi_{E_2};$$

(v) $$E_1\cup E_2=\left(\max\left\{\theta_{E_1},\ \theta_{E_2}\right\},\ \min\left\{\phi_{E_1},\ \phi_{E_2}\right\}\right);$$

(vi) $$E_1\cap E_2=\left(\min\left\{\theta_{E_1},\ \theta_{E_2}\right\},\ \max\left\{\phi_{E_1},\ \phi_{E_2}\right\}\right);$$

(vii) $\lambda E = \left(\left(1 - \left(1 - \theta_E \right)^{\lambda} \right)^{1/3},\ \phi_E^{\lambda} \right),\ \lambda > 0;$

(viii) $E^{\lambda} = \left(\theta_E^{\lambda}, \left(1 - \left(1 - \phi_E \right)^{\lambda} \right)^{1/3} \right), \lambda > 0;$

(ix) $E^{c} = \left(\theta_E, \phi_E \right)^{c} = \left(\phi_E, \theta_E \right).$

Definition 3 (Senapati & Yager, 2019a): Let $E = \left(\theta_E, \phi_E \right)$ be a FFN, then Score Function (SF) of the FFN is defined as $S(E) = \theta_E^3 - \phi_E^3$, $S(E) \in [-1,\ 1]$

Definition 4 (Senapati &Yager, 2019b)**:** Let $E = \left(\theta_E, \phi_E \right)$ be a FFN, then the accuracy function of the FFN is defined as $H(E) = \theta_E^3 + \phi_E^3$ where $0 \le H(E) \le 1.$

Definition 5 (Senapati & Yager, 2019b)**:** Let $E_1 = \left(\theta_{E_1}, \phi_{E_1} \right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2} \right)$ be two Fermatean fuzzy numbers (FFNs). The relationship between their score functions is described below.

(i) If $S(E_1) < S(E_2)$ then $E_1 < E_2$;

(ii) If $S(E_1) > S(E_2)$ then $E_1 > E_2$;

(iii) If $S(E_1) = S(E_2)$ then the accuracy of each FFN E_1 and E_2 is computed.

(a) If $H(E_1) < H(E_2)$ then $E_1 < F_2$;

(b) If $H(E_1) > H(E_2)$ then $E_1 > E_2$;

(c) If $H(E_1) = H(E_2)$ then $E_1 = E_2$.

Definition 6: Let $E = \left(\theta_E, \phi_E \right)$, $E_1 = \left(\theta_{E_1}, \phi_{E_1} \right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2} \right)$ represent three Fermatean fuzzy numbers (FFNs). The following operational laws are valid:

(i) $E_1 \oplus E_2 = E_2 \oplus E_1$

(ii) $E_1 \otimes E_2 = E_2 \otimes E_1$

(iii) $\mu \left(E_1 \oplus E_2 \right) = \mu E_1 \oplus E_2,\ \ \mu > 0;$

(iv) $\left(\mu_1 \oplus \mu_2 \right) E = \mu_1 E \oplus \mu_2 E,\ \ \ \mu_1, \mu_2 > 0;$

(v) $\left(E_1 \oplus E_2 \right)^{\mu} = E_1^{\mu} \oplus E_2^{\mu},\ \ \mu > 0;$

(vi) $E^{\mu_1} \oplus E^{\mu_2} = E^{\mu_1 + \mu_2},\ \mu_1, \mu_2 > 0.$

Definition 7: Fermatean fuzzy weighted aggregation operator:

Let $E_j = (\theta_{F_j}, \phi_{F_j}), (j = 1, 2, ..., n)$ be n number of FFNs $w_j (j = 1, 2, 3, ..., n)$ be respective weights of the Fermatean Fuzzy Numbers such that $0 \le w_j \le 0$ and $\sum_{j=1}^{n} w_j = 1$, then the Fermatean Fuzzy Weighted Aggregation (FFWA) operator can be defined as follows.

$$FFWA = \left(\frac{\sum_{j=1}^{n} w_j \theta_{F_j}}{\sum_{j=1}^{n} w_j}, \frac{\sum_{j=1}^{n} w_j \phi_{F_j}}{\sum_{j=1}^{n} w_j} \right)$$

FFWA operator is a mathematical construct used in FL and FST to aggregate multiple inputs while considering their degrees of membership and associated weights.

2.2 Properties of FFWA operators

The properties of boundness, monotonicity, and idempotency of the FFWA operator are as follows:

- Boundness: The FFWA operator is bounded within the range of the elements being aggregated. If all elements in A are bounded within a certain interval, then the FFWA operator will also produce a result within that interval. Mathematically,

  If $\min(a_i) \le a_i \le \max(a_i)$ for all i, then $\min(a_i) \le FFWA(A,W) \le \max(a_i)$.

- Monotonicity: The FFWA operator is monotonic if increasing the membership values of elements in A or increasing their corresponding weights in W leads to an increase in the aggregated value. Mathematically,

  If $a_i \le b_i$ and $w_i \le v_i$ for all i then FFWA(A, W) $\le$ FFWA (B, V).

- Idempotency: The FFWA operator is idempotent if applying the operator multiple times to the same set of inputs produces the same result. Mathematically

  FFWA(A,W) = FFWA(FFWA (A, W), W).

These properties make the FFWA operator useful for aggregating fuzzy information while preserving certain desirable mathematical characteristics. However, it's important to note that the exact properties of the FFWA operator can vary depending on specific definitions and contexts in which it is applied.

Definition 8: The distance between two Fermatean fuzzy sets can be defined as:

Let, $E_1 = \left(\theta_{E_1}, \phi_{E_1}\right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2}\right)$ are two FFSs. Then the distance of the FFSs can be described below.

(i) Hamming Fermatean fuzzy distance of $E_1 = \left(\theta_{E_1}, \phi_{E_1}\right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2}\right)$ is defined as follows:

$$D(E_1, E_2) = \frac{1}{2}\left[\left|\theta_{E_1}^3 - \theta_{E_2}^3\right| + \left|\phi_{E_1}^3 - \phi_{E_2}^3\right| + \left|\pi_{E_1}^3 - \pi_{E_2}^3\right|\right]$$

(ii) Euclidean Fermatean fuzzy distance of $E_1 = \left(\theta_{E_1}, \phi_{E_1}\right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2}\right)$ is defined as follows:

$$D(E_1, E_2) = \left(\frac{1}{2}\left[\left(\theta_{E_1}^3 - \theta_{E_2}^3\right)^2 + \left(\phi_{E_1}^3 - \phi_{E_2}^3\right)^2 + \left(\pi_{E_1}^3 - \pi_{E_2}^3\right)^2\right]\right)^{1/2}$$

(iii) Generalized Fermatean fuzzy distance of $E_1 = \left(\theta_{E_1}, \phi_{E_1}\right)$ and $E_2 = \left(\theta_{E_2}, \phi_{E_2}\right)$ is defined in the following way:

$$D(E_1, E_2) = \left[\frac{1}{2}\left\{\left(\theta_{E_1}^3 - \theta_{E_2}^3\right)^{\gamma} + \left(\phi_{E_1}^3 - \phi_{E_2}^3\right)^{\gamma} + \left(\pi_{E_1}^3 - \pi_{E_2}^3\right)^{\gamma}\right\}\right]^{1/\lambda}$$

3. The Proposed Fermatean Fuzzy Based Heterogeneous Group TOPSIS Algorithm

In this section, we introduce a new heterogeneous group TOPSIS algorithm based on Fermatean fuzzy sets. The steps of the algorithm are outlined below.

Step 1: *Construction of expert committee, identification of criteria and enlisting the feasible alternatives:* The process involves assembling a heterogeneous expert group consisting of top DMs, identifying key criteria by the experts, and compiling a list of available and feasible alternatives by the expert group. Denote the experts by $\left(E_1,...,E_k,...,E_p\right)^T$, the set of significant criteria by $\left(C_1,...,C_j,...,C_n\right)$ and the set of alternatives by $\left\{A_1,...,A_i,...,A_m\right\}^T$. The symbol (^{T}) implies transpose matrix.

Step 2: *Setting linguistic variables and Fermatean fuzzy sets:* Set the grades of linguistic variables for the estimation of performance ratings of alternatives and judgement of criteria weights.

In the context of a Multiple Criteria Decision Making (MCDM) algorithm, linguistic variables and Fermatean fuzzy sets play a crucial role in handling subjective information and uncertainty in decision-making processes. The background beyond the using linguistic variables is as follows:

Subjectivity in Decision Making: Decision-making often involves subjective judgments and preferences that are difficult to quantify precisely. For example, when evaluating alternatives or assigning weights to decision criteria, decision-makers may express their preferences using qualitative terms like "very good," "moderate," or "poor."

Uncertainty and Ambiguity: In many real-world decision scenarios, there is inherent uncertainty and ambiguity due to incomplete information, vague preferences, or conflicting criteria. Linguistic variables help in capturing this uncertainty by allowing decision-makers to express their preferences in a more flexible and interpretable manner.

Fuzzy Set Theory: Fuzzy set theory provides a mathematical framework for dealing with uncertainty and vagueness. It extends traditional set theory by allowing elements to belong to sets to a degree, rather than strictly being either inside or outside the set. Fermatean fuzzy sets are a specific type of fuzzy set defined by Fermatean semantics, which emphasizes the gradual transition between membership degrees.

The necessity beyond the using linguistic variables is as follows::

Expressiveness: Linguistic variables allow decision-makers to express their preferences and judgments in natural language terms, making the decision-making process more intuitive and accessible. This enables stakeholders with varying expertise levels to participate in the decision-making process effectively.

Handling Uncertainty: Fermatrean fuzzy based MCDM technique is an effective tool in modelling and handling uncertainty and ambiguity in making decision. This is particularly important when dealing with imprecise or incomplete information, which is common in many real-world decision scenarios.

Flexibility and Adaptability: Linguistic variables provide a flexible framework that can accommodate diverse preferences and viewpoints. Decision-makers can adjust the linguistic terms and their associated grades to reflect their subjective assessments accurately. This adaptability enhances the robustness and applicability of the MCDM algorithm across different decision contexts.

The use of linguistic variables and Fermatean fuzzy sets in Step 2 of the proposed MCDM algorithm is essential for capturing subjective preferences, handling uncertainty, and providing a flexible and intuitive decision-making framework. These linguistic representations enable decision-makers to express their judgments in natural language terms and facilitate the effective modelling of uncertainty and ambiguity in decision processes.

Step 3: *Assessment of expert weight:* Each expert is assessed with respect to some meaningful factors (attributes). A matrix for measuring expert weight is constructed with the help the expert group.

$$M_{ew} = \begin{matrix} \\ E_1 \\ ... \\ E_k \\ ... \\ E_p \end{matrix} \begin{matrix} \mathrm{F_1} \; ... \; \mathrm{F_j} \; ... \; \mathrm{F_q} \\ \begin{bmatrix} x_{11} & ... & x_{1j} & ... & x_{1q} \\ ... & ... & ... & ... & ... \\ x_{k1} & ... & x_{kj} & ... & x_{kq} \\ ... & ... & ... & ... & ... \\ x_{p1} & ... & x_{pj} & ... & x_{pq} \end{bmatrix} \end{matrix} = \begin{matrix} E_1 \\ ... \\ E_k \\ ... \\ E_p \end{matrix} \begin{bmatrix} r_{11} & ... & r_{1j} & ... & r_{1q} \\ ... & ... & ... & ... & ... \\ r_{k1} & ... & r_{kj} & ... & r_{1q} \\ ... & ... & ... & ... & ... \\ r_{p1} & ... & r_{pj} & ... & r_{1q} \end{bmatrix} = \begin{bmatrix} GM_1 \\ ... \\ GM_k \\ ... \\ GM_p \end{bmatrix} = \begin{bmatrix} w_{e1} \\ ... \\ w_{ek} \\ ... \\ w_{ep} \end{bmatrix} \quad (1)$$

Where, x_{kj} denotes the assessment value of expert E_k under the factor $\mathrm{F_j}$. q is the number of factors considered for assessment of experts.

where, $r_{kj} = \frac{w_{F_J} * x_{kj}}{\sum_{k=1}^{p} x_{kj}}$, w_{F_J} is weight of factor F_j. $GM_k = \left(\prod_{j=1}^{q} r_{kj}\right)^{1/q}$, $w_{ek} = \frac{GM_k}{\sum_{k=1}^{p} GM_k}$, and $\sum_{k=1}^{p} w_{ek} = 1$.

w_{ek} is the weight of expert E_k.

Step 4: *Formation of criteria weight matrix using linguistic variables:* The importance weight of each criterion is judged by each expert with his/her own experience, view, analysis and perception using prescribed set of linguistic variables. A weight matrix is constructed by the experts as follows.

$$M_{CW(l)}^{k} = \left[v_{j(l)}^{k}\right]_{p\times n} = \begin{array}{c} \\ E_1 \\ \cdots \\ E_k \\ \cdots \\ E_p \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{C_1} & \cdots & \mathrm{C_j} & \cdots & \mathrm{C_n} \end{array} \\ \begin{bmatrix} v_{1(l)}^{1} & \cdots & v_{j(l)}^{1} & \cdots & v_{n(l)}^{1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{1(l)}^{k} & \cdots & v_{j(l)}^{k} & \cdots & v_{n(l)}^{k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{1(l)}^{p} & \cdots & v_{j(l)}^{p} & \cdots & v_{n(l)}^{p} \end{bmatrix} \end{array} \tag{2}$$

$v_{j(l)}^{k}$ is the weight of criterion $\mathrm{C_j}$ in terms of linguistic variable as assessed by the expert E_k.

Step 5: *Determination of criteria weight matrix in terms of Fermatean fuzzy set:* Calculate the Fermatean fuzzy importance weight for each criterion $C_j\,(j = 1,2,3,...,n)$ under consideration with the help of each expert E_k $(k = 1,2,3,..., p)$.

$$W = \left[\beta_{ij}\right]_{p\times n} = \begin{array}{c} \\ E_1 \\ \cdots \\ E_k \\ \cdots \\ E_p \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_j & \cdots & C_n \end{array} \\ \begin{bmatrix} \langle \tau_{11}, \rho_{11} \rangle & \langle \cdots \rangle & \langle \tau_{1j}, \rho_{1j} \rangle & \langle \cdots \rangle & \langle \tau_{1n}, \rho_{1n} \rangle \\ \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle \\ \langle \tau_{k1}, \rho_{k1} \rangle & \langle \cdots \rangle & \langle \tau_{kj}, \rho_{kj} \rangle & \langle \cdots \rangle & \langle \tau_{kn}, \rho_{kn} \rangle \\ \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle & \langle \cdots \rangle \\ \langle \tau_{p1}, \rho_{p1} \rangle & \langle \cdots \rangle & \langle \tau_{pj}, \rho_{pj} \rangle & \langle \cdots \rangle & \langle \tau_{pn}, \rho_{pn} \rangle \end{bmatrix}_{p\times n} \end{array} \tag{3}$$

Here $\left\langle \tau_{kj}, \rho_{kj} \right\rangle$ is the Fermatean fuzzy set of linguistic variable $v_{j(l)}^{k}$ representing the weight of the criterion C_j and assessed by expert E_k.

Step 6: *Computation of criteria weight by the proposed method***:** Each expert estimates the importance weight of each criterion in terms of Fermatean fuzzy set. Aggregate all the Fermatean fuzzy weights of a particular criterion assessed by different experts $(e_k \,|\, k = 1,2,3...p)$ and transform them into a single combined Fermatean fuzzy weight. Apply the Fermatean fuzzy weighted average (FFWA) operator to combine all the weight matrices formed by the group of experts into a single combined weight matrix.

$$\bar{W} = \left[\bar{\beta}_{ij}\right]_{1\times n} = \left\langle \frac{1}{p}\sum_{k=1}^{p}\left(w_{ek} * \tau_{kj}\right), \frac{1}{p}\sum_{k=1}^{p}\left(w_{ek} * \rho_{kj}\right) \right\rangle_{1\times n} = \left\langle \bar{\tau}_j, \bar{\rho}_j \right\rangle_{1\times n} \tag{4}$$

$$w_j^d = \frac{\left[\left(\tau_j^3 - \rho_j^3\right)\left(\tau_j^3 + \rho_j^3\right)\right]^2}{\sum_{j=1}^{n}\left[\left(\tau_j^3 - \rho_j^3\right)\left(\tau_j^3 + \rho_j^3\right)\right]^2} \tag{5}$$

w_j^d is the coefficient of weight measured by experts' knowledge, intuition and view for criterion C_j.

Step 7: *computation of criteria weight by Fermatean fuzzy Entropy technique:* Compute the coefficient of importance weight of each criterion by the application of Fermatean fuzzy entropy weight measuring method. To estimate the importance weight of criterion using FF entropy method following steps are followed.

Calculation of e_{ij} value $= 1 - \left[\left(\overline{\theta}_{ij}^{3}(x) - \overline{\phi}_{ij}^{3}(x) \right) \left(\overline{\theta}_{ij}^{3}(x) + \overline{\phi}_{ij}^{3}(x) \right) \right]^{2}$ (6)

Calculation of the entropy $\tilde{e}_j = \frac{1}{m} \sum_{i=1}^{m} e_{ij}$ (7)

Calculation of the criteria weights by $w_j^e = \frac{(1-\overline{e}_j)}{\sum_{j=1}^{n}(1-\overline{e}_j)}$ (8)

Step 8: *Computation of combined criteria weights* (w_j) using the following equation

$$w_j = \xi * (w_j^d) + (1-\xi) * w_j^e \quad (9)$$

where, w_j is the combined weight coefficient for the criterion C_j . w_j^d is weight of criterion C_j computed by the proposed method based on Fermatean fuzzy decision makers' (expert) preference. w_j^e is weight of criterion C_j by the Fermatean fuzzy entropy weighing method. Here, ξ and $(1-\xi)$ are the coefficients of experts' attitude (preference) towards the combined weight components.

Step 9: *Formation of decision matrices:* Each alternative is evaluated by each expert against the criteria in terms of predefined linguistic variables. Each expert constructs a decision matrix in the following manner.

$$M_{D(l)}^{k} = \left[\mu_{ij(l)}^{k} \right]_{m \times n} = \begin{array}{c} \\ A_1 \\ \cdots \\ A_i \\ \cdots \\ A_m \end{array} \begin{array}{c} \begin{matrix} C_1 & \cdots & C_j & \cdots & C_n \end{matrix} \\ \begin{bmatrix} \mu_{11(l)}^{k} & \cdots & \mu_{1j(l)}^{k} & \cdots & \mu_{1n(l)}^{k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{i1(l)}^{k} & \cdots & \mu_{ij(l)}^{k} & \cdots & \mu_{in(l)}^{k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{m1(l)}^{k} & \cdots & \mu_{mj(l)}^{k} & \cdots & \mu_{mn(l)}^{k} \end{bmatrix} \end{array} \quad (10)$$

The linguistic performance rating ($\mu_{ij(l)}^{k}$) of each alternative A_i is evaluated based on specific criteria, as assessed by the experts. The decision matrix is constructed according to the number of experts participating in the decision-making process. In this context, the variables *(l)* represent linguistic terms.

Step 10: Constructing the Fermatean fuzzy (FF) decision matrix: Each expert (denoted as k=1, 2, 3, ..., p) evaluates every alternative $A_i (i = 1,2,3...m)$ with respect to each criterion $C_j (j = 1,2,3...n)$. These assessments collectively form the Fermatean fuzzy decision matrix M^k , which is organized as follows.

$$M^k=\left[\alpha_{ij}^k\right]_{m\times n}=\begin{matrix} \\ A_1\\ \dots\\ A_i\\ \dots\\ A_m\end{matrix}\begin{matrix} C_1 & \dots & C_j & \dots & C_n\\ \left[\begin{matrix}\left\langle\theta_{11}^k(x),\ \phi_{11}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{1j}^k(x),\phi_{1j}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{1n}^k(x),\phi_{1n}^k(x)\right\rangle\\ \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle\\ \left\langle\theta_{i1}^k(x),\ \phi_{i1}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{ij}^k(x),\phi_{ij}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{in}^k(x),\phi_{in}^k(x)\right\rangle\\ \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle\\ \left\langle\theta(x)_{m1}^k,\ \phi_{m1}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{mj}^k(x),\phi_{mj}^k(x)\right\rangle & \langle\dots\rangle & \left\langle\theta_{mn}^k(x),\phi_{mn}^k(x)\right\rangle\end{matrix}\right]_{m\times n}\end{matrix} \quad (11)$$

In the decision matrix above, the Fermatean fuzzy set is represented by the symbol $\alpha_{ij}^k=\left\langle\theta_{ij}^k(x),\phi_{ij}^k(x)\right\rangle$, while the degree of membership and the degree of non-membership of each element are denoted by $\theta_{ij}^k(x)$ and $\phi_{ij}^k(x)$ accordingly.

Step 11: *Normalizing the Fermatean fuzzy decision matrices:* If the Fermatean fuzzy decision matrix includes both benefit and cost criteria, the performance ratings under cost criteria need to be normalized. The following equation is proposed for the normalization process.

$$\alpha_{ij}^{kN}=\left\langle\theta_{ij}^{kN}(x),\phi_{ij}^{kN}\right\rangle=\begin{cases}\left\langle\theta_{ij}^k(x),\phi_{ij}^k\right\rangle, j\in\text{Benefit criteria}\\ \left\langle\theta_{ij}^k(x),\phi_{ij}^k\right\rangle^c=\left\langle\phi_{ij}^k(x),\theta_{ij}^k\right\rangle, j\in\text{Cost criteria}\end{cases} \quad (12)$$

Where, $i=1,2...m;\ j=1,2...n;$ and $k=1,2...p.$ It should be noted that the Fermatean performance ratings under benefit criteria remain unchanged even after the normalization process.

Step 12: *Aggregating Fermatean fuzzy decision matrices (AFFDM):* The Fermatean fuzzy decision matrices are aggregated by applying the expert weights $(e_k|k=1,2...p)$ and combined into a single matrix. To achieve this, the Fermatean fuzzy weighted average (FFWA) operator is used to merge the matrices contributed by each individual expert.

$$M=\left[\alpha_{ij}\right]_{m\times n}=\left[\sum_{k=1}^{p}w_{ek}*\alpha_{ij}^{kN}\right]_{m\times n}=\begin{matrix} \\ A_1\\ \dots\\ A_i\\ \dots\\ A_m\end{matrix}\begin{matrix} C_1 & \dots & C_j & \dots & C_n\\ \left[\begin{matrix}\langle\bar\theta_{11}(x),\bar\phi_{11}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{1j}(x),\bar\phi_{1j}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{1n}(x),\bar\phi_{1n}(x)\rangle\\ \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle\\ \langle\bar\theta_{i1}(x),\bar\phi_{i1}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{ij}(x),\bar\phi_{ij}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{in}(x),\bar\phi_{in}(x)\rangle\\ \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle & \langle\dots\rangle\\ \langle\bar\theta(x)_{m1},\bar\phi_{m1}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{mj}(x),\bar\phi_{mj}(x)\rangle & \langle\dots\rangle & \langle\bar\theta_{mn}(x),\bar\phi_{mn}(x)\rangle\end{matrix}\right]_{m\times n}\end{matrix} \quad (13)$$

Step 13: *Identification of the Fermatean fuzzy positive ideal solution (FFPIS) and Fermatean fuzzy negative ideal solution (FFNIS):* The FFPIS and FFNIS are determined using score functions. The FFPIS is represented as I^+, while the FFNIS is denoted as I^-. These can be calculated by applying the corresponding equations.

Step 13: *Determination of the Fermatean fuzzy positive ideal solution (FFPIS) and Fermatean fuzzy negative ideal solution (FFNIS)*: FFPIS and FFNIS are determined based on score functions. FFPIS is denoted by I^+ and FFNIS is denoted by I^- which can be determined by the application of the respective equation.

$$I^{+}=\begin{cases}\max\limits_{i}\left(score(C_j(A_i))\right), j=1,2,...,n; C_j \in \text{benefit criteria}\\ \min\limits_{i}\left(score(C_j(A_i))\right), j=1,2,...,n; C_j \in \text{cost criteria}\end{cases}$$

$$=\left\{s_1^{+},...,s_j^{+},....,s_n^{+}\right\} \tag{14}$$

$$=\left\{\left\langle \tau_1^{+},\rho_1^{+}\right\rangle,...,\left\langle \tau_j^{+},\rho_j^{+}\right\rangle,....,\left\langle \tau_n^{+},\rho_n^{+}\right\rangle\right\}$$

$$I^{-}=\begin{cases}\min\limits_{i}\left\langle score\ (C_j(A_i))\right\rangle, j=1,2...n;\ \ C_j \in \text{Benefit criteria}\\ \max\limits_{i}\left\langle score\ (C_j(A_i))\right\rangle, j=1,2...n;\ \ C_j \in \text{Cost criteria}\end{cases}$$

$$=\left\{s_1^{-},...,s_j^{-},...,s_n^{-}\right\} \tag{15}$$

$$=\left\{\left\langle \tau_1^{-},\rho_1^{-}\right\rangle,...,\left\langle \tau_j^{-},\rho_j^{-}\right\rangle,...,\left\langle \tau_n^{-},\rho_n^{-}\right\rangle\right\}$$

Step 14: *Calculation of the weighted Euclidean distances between alternatives and the Fermatean fuzzy ideal solutions:* The weighted Euclidean distance between an alternative and the Fermatean fuzzy positive ideal solution (FFPIS) is calculated using the following equation.

$$D\left(A_i,I^{+}\right)=\sum_{j=1}^{n} w_j d\left(C_j\left(A_i\right),C_j\left(I^{+}\right)\right)$$

$$=\sum_{j=1}^{n} w_j\left(\frac{1}{2}\left[\left(\tau_{ij}^{3}-(\tau_j^{+})^{3}\right)^{2}+\left(\rho_{ij}^{3}-(\rho_j^{+})^{3}\right)^{2}+\left(\pi_{ij}^{3}-(\pi_j^{+})^{3}\right)^{2}\right]\right)^{1/2} \tag{16}$$

Where, $j=1,2...n;$ The lower value of $D(A_i,I^{+})$ ensures the better alternative, since lower value indicates that the separation of the alternative is closer to the FFPIS.

Step 15: *Calculation of the weighted Euclidean distances between alternatives and the Fermatean fuzzy negative ideal solution (FFNIS):* The weighted Euclidean distance between an alternative and the FFNIS is determined using the following equation.

$$D\left(A_i,I^{-}\right)=\sum_{j=1}^{n} w_j d\left(C_j\left(A_i\right),C_j\left(I^{-}\right)\right)$$

$$=\sum_{j=1}^{n} w_j\left(\frac{1}{2}\left[\left(\tau_{ij}^{3}-(\tau_j^{-})^{3}\right)^{2}+\left(\rho_{ij}^{3}-(\rho_j^{-})^{3}\right)^{2}+\left(\pi_{ij}^{3}-(\pi_j^{-})^{3}\right)^{2}\right]\right)^{1/2} \tag{17}$$

Where $j=1,2...n;$ The higher value of $D(A_i,I^{-})$ ensures the superior alternative, since higher value indicates that the separation of the alternative is closer to the FFPIS.

The minimum $D(A_i,I^{+})$, $i=1,2...m$ is denoted as $\min\limits_{i} D(A_i,I^{+})=D_{\min}(A_i,I^{+})$ and the maximum $D(A_i,I^{-})$, $i=1,2...m$ is denoted by $\max\limits_{i} D(A_i,I^{+})=D_{\max}(A_i,I^{-})$

Step 16: *Proximity Coefficient:* Proximity Coefficient of an alternative A_i can be measured with the proposed Eq. (18).

$$PC\left(A_i\right)=\frac{D(A_i,I^-)}{D_{\max}(A_i,I^-)+D_{\min}(A_i,I^-)}-\frac{D(A_i,I^+)}{D_{\max}(A_i,I^+)+D_{\min}(A_i,I^+)} \tag{18}$$

The alternatives are then ranked in descending order based on their proximity coefficients. The alternative with the highest proximity coefficient is identified as the best alternative, while the one with the lowest proximity coefficient is considered the worst alternative.

Step 17: *Closeness Coefficients:* In the FF-TOPSIS method, the closeness coefficient of an alternative is calculated as outlined in the following Eq. (19).

$$CC\left(A_i\right)=\frac{D(A_i,I^-)}{D(A_i,I^-)+D(A_i,I^+)} \tag{19}$$

Similarly, the alternatives are ranked in descending order based on their closeness coefficients. The alternative with the highest closeness coefficient is considered the best, while the one with the lowest value is regarded as the worst.

Step 18: Calculation of the Combined Selection Index (CSI): The following equation is proposed to calculate the combined selection index (CSI) for an alternative:

$$CSI\left(A_i\right)=\lambda * PC\left(A_i\right)+\left(1-\lambda\right)* CC\left(A_i\right) \tag{20}$$

Where the value of λ is constrained by $0 \le \lambda \le 1$. The alternatives are then ranked based on their CSI values in descending order. The alternative with the highest CSI is selected as the best.

Fig.1 illustrates the framework of the proposed methodology, clearly showing the various steps and their sequence in the flow diagram. Fig.2 depicts the inputs and outputs associated with the developed method. As seen, the inputs include expert group opinions, expert weight matrix, criteria set, criteria weight matrix, feasible alternatives, and decision matrices. The method then processes these inputs to provide outputs such as the ranking order of alternatives, informed decision-making, identification of the most influential experts, and determination of the most important criteria.

The above described Fermatean fuzzy based algorithm is numerically illustrated in the following section by solving a decision making problem on warehouse location selection.

4. Numerical Example with Illustration

In section 4, we demonstrate the practical application of our proposed framework through a real-world example of Warehouse Location Selection (WLS). This case study is designed to evaluate the effectiveness and feasibility of our method in a practical scenario. We begin by outlining the problem.

4.1 Warehouse Location Selection Problem

An automobile company, based in a South Asian country, is looking to expand its operations in order to increase its market share. As part of their strategy to grow, the company plans to establish a new plant and is in the process of selecting an optimal location for a warehouse to serve the region's substantial market potential. To achieve this goal, the company’s management forms a team composed of four experts from key departments within the organization: finance, logistics, production, and marketing. These departments are integral to the company's operations, and the selected experts bring over ten years of experience in their respective fields, offering a well-rounded approach to decision-making.

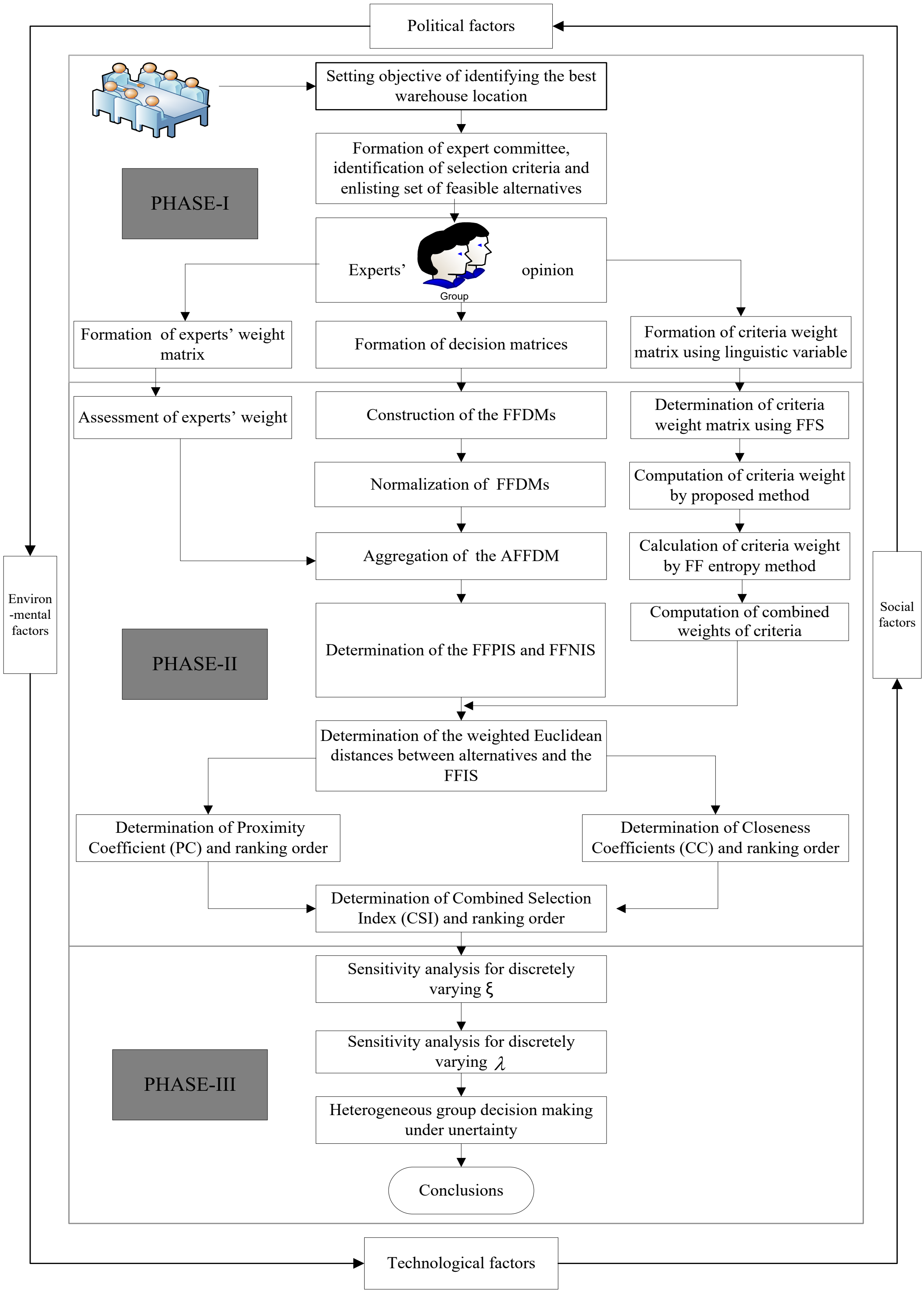

Fig.1. Framework of the proposed methodology

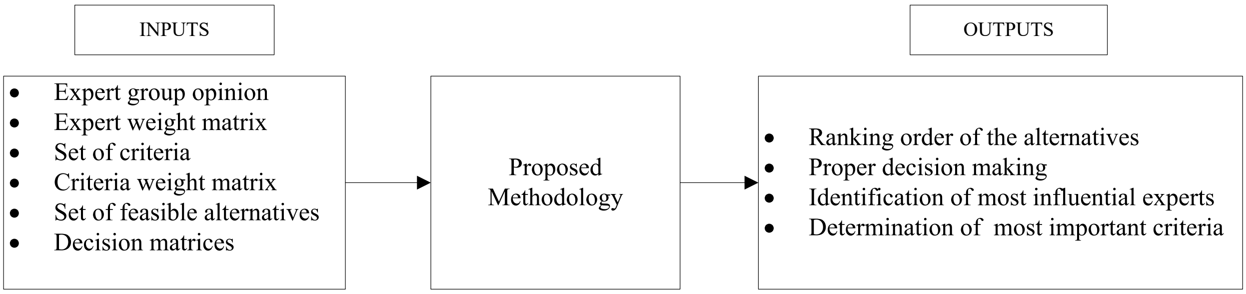

Fig. 2. Inputs and outputs of the proposed method.

Our proposed algorithm focuses on eliciting the preferences of these experts regarding various criteria relevant to the selection of a warehouse location. To this end, the proficiency levels of the experts across different criteria are assessed. The expert group devises a number of questionnaire following an extensive literature survey and fruitful discussions with top-level management to address the strategic requirements. These questionnaires undergo thorough scrutiny by supervisors and managers within the organization, leading to the identification of a feasible set of four cities in the Indian state of West Bengal as potential warehouse locations: Haldia (A_1), Durgapur (A_2), Asansol (A_3), and Bardhaman (A_4). Fig. 3 Shows the Geographic positions of the four alternatives Warehouse Locations viz. Haldia, Durgapur, Asansol and Bardhaman in the Indian state of West Bengal.

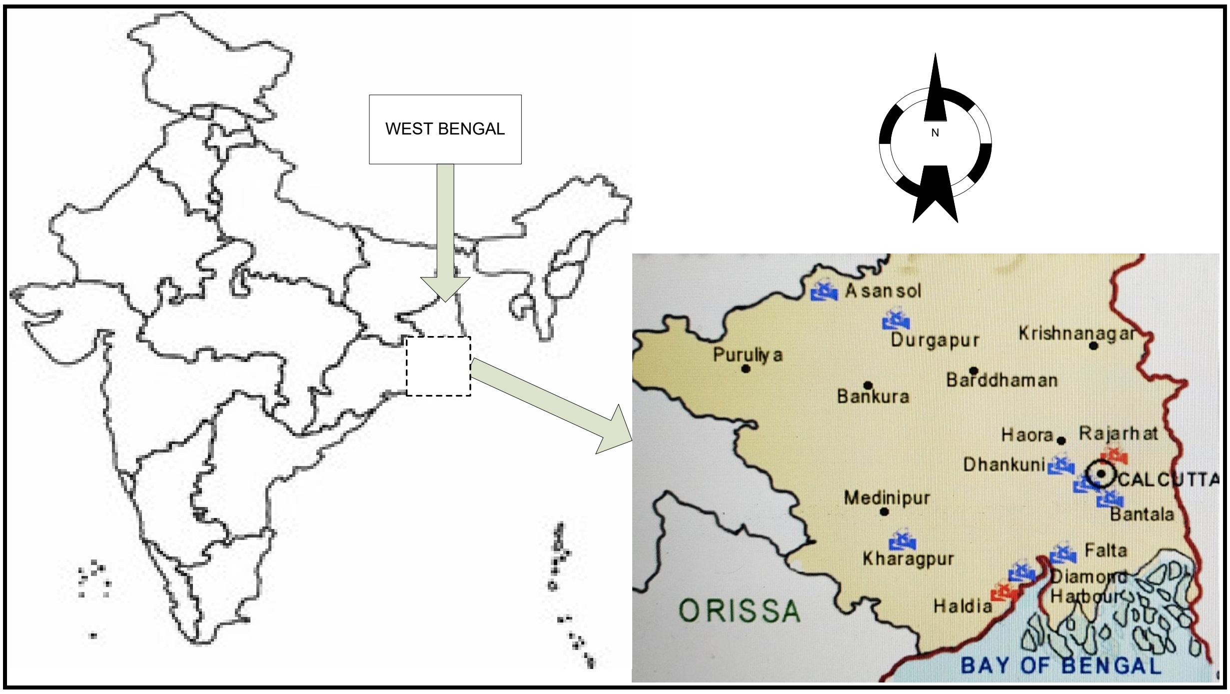

Fig. 3. Geographic positions of the four alternatives Haldia, Durgapur, Asansol and Bardhaman in the Indian state of West Bengal (Map not to scale. It is only for illustration purpose).

4.1.1 Alternative Warehouse Locations

Each of the proposed warehouse locations serves as a strategic option for catering to a densely populated demographic with medium to high purchasing power, primarily located in the vicinity of Kolkata. Let's delve into the brief descriptions of each alternative location:

Haldia (A_1):

- Location: Haldia is a crucial industrial port city situated in the East Midnapur district of West Bengal.
- Characteristics: It serves as a vital river port and industrial hub, strategically positioned approximately 124 km away from Kolkata, near the mouth of the Hooghly River.

- Significance: Haldia's strategic location provides access to maritime transportation routes, making it an attractive choice for warehousing operations that require efficient import and export logistics. Its industrial infrastructure further enhances its suitability for accommodating storage and distribution facilities.

Durgapur (A_2):

Location: Durgapur is a significant industrial city located in the Paschim Bardhaman district of West Bengal.

Characteristics: As the fourth largest urban agglomeration in tier-II India, Durgapur holds a distinct position as the only city in eastern India with an operational dry dock.

Significance: Durgapur's industrial prowess, coupled with its unique infrastructure capabilities, makes it an appealing option for warehousing operations targeting the eastern region of India. Its status as the 'Ruhr' of India underscores its importance as an industrial and logistical center.

Asansol (A_3):

Location: Asansol is a tier II metropolitan city situated in West Bengal, recognized as the most populous and second-largest city in the state.

Characteristics: Serving as the headquarters of the Paschim Bardhaman district, Asansol is renowned for its rapid urbanization and growth trajectory.

Significance: Asansol's demographic profile and urban development make it a promising location for establishing warehousing facilities catering to the growing consumer demand in the region. Its strategic position within West Bengal further enhances its accessibility and logistical advantages.

Bardhaman (A_4):

Location: Bardhaman is a municipal city and district headquarters located in the Purba Bardhaman district of West Bengal.

Characteristics: With a population exceeding four lakhs and a substantial area, Bardhaman has served as a district capital since the British colonial era.

Significance: Bardhaman's historical and administrative significance, combined with its demographic profile, presents opportunities for establishing warehousing infrastructure to serve the local market and adjacent regions. Its status as a district headquarters further adds to its attractiveness as a potential logistics hub.

In summary, each alternative warehouse location offers unique advantages in terms of proximity to Kolkata, demographic characteristics, industrial infrastructure, and logistical connectivity, making them viable options for fulfilling the warehousing needs of the targeted consumer base.

4.1.2 Selection Criteria

The expert group identifies five critical selection criteria essential for evaluating warehouse locations, defined as follows:

Space Availability (C_1): Adequate space availability is essential to comply with government regulations concerning warehouse locations.

Transportation Facility (C_2): Availability of comprehensive transportation facilities ensures timely delivery of goods to customers, with considerations for various modes of transportation such as road, rail, air, and water.

Market Accessibility (C_3): Market accessibility pertains to the proximity of supplier bases, manufacturing plants, and consumer markets to the selected warehouse location, facilitating efficient product distribution.

Workforce Availability (C_4): Adequate availability of skilled labor is crucial for warehouse operations, ensuring timely and efficient fulfilment of tasks.

Cost (C5): Cost considerations encompass both monetary and non-monetary sacrifices associated with resource allocation, with the aim of optimizing total costs associated with each warehouse location alternative.

In the subsequent sections, we elaborate on our proposed Fermatean fuzzy-based decision-making algorithm and provide a mathematical problem to illustrate its application in warehouse location selection. We also conduct sensitivity analysis to assess the robustness of our technique in varying decision-making contexts. Finally, we conclude with essential remarks and outline potential avenues for future research.

4.2 Calculation and Discussions

In this section the step by step description of the calculation procedure of the proposed approach has been furnished. The associated result and its significance have also been discussed for the purpose of explanation.

Step 1: In this phase of the problem, a diverse group of four experts, labeled as E_1, E_2, E_3, and E_4, has been assembled. These experts are responsible for selecting five key criteria, which include C_1: Market accessibility, C_2: Transportation infrastructure,

C_3: Space availability, C_4: Workforce availability, and C_5: Cost. Additionally, the expert team identifies four viable alternatives, represented as A_1: Haldia, A_2: Durgapur, A_3: Asansol, and A_4: Bardhaman.

Step 2: This step proposes a set of ten linguistic variables to assess the performance ratings of alternative warehouse locations and the relative importance of the criteria. A standardized set of linguistic variables, along with their corresponding Fermatean fuzzy sets—comprising both membership and non-membership degrees—are provided in Table 1 for evaluating the performance of alternatives and the weights of the criteria. The weights assigned to each expert are calculated using the formula in Eq.(1).

Table 1
Linguistic terms for evaluating alternatives and criteria weights

Linguistic Terms for Performance Ratings of Alternatives	Linguistic Terms for Weights of Criteria	Fermatean Fuzzy Numbers (FFNs)
Extremely High (EH)	Extremely Important (EI)	(0.95, 0.15)
Very High (VH)	Very Important (VI)	(0.90, 0.20)
High (H)	Important (I)	(0.80, 0.30)
Fairly High (FH)	Fairly Important (FI)	(0.70, 0.40)
Slightly High (SH)	Slightly Important (SI)	(0.60, 0.50)
Slightly Low (SL)	Slightly Unimportant (SU)	(0.50, 0.60)
Fairly Low (FL)	Fairly Unimportant (FU)	(0.40, 0.70)
Low (L)	Unimportant (U)	(0.30, 0.80)
Very Low (VL)	Very Unimportant (VU)	(0.20, 0.90)
Extremely Low (EL)	Extremely Unimportant (EU)	(0.15, 0.95)

Linguistic Terms for Assessing Performance Ratings of Alternatives: This section lists a series of linguistic terms that describe the performance levels of various alternatives. These terms range from "Extremely High" to "Extremely Low," representing different degrees of performance or desirability of each option.

Linguistic Terms for Estimating the Weight of Criteria: This section outlines the linguistic terms used to evaluate the importance or weight of various criteria during decision-making. The terms span from "Extremely Important" to "Extremely Unimportant," signifying how significant each criterion is considered in the decision process.

Fermatean Fuzzy Numbers (FFNs): This section associates each linguistic term with its corresponding FFNs which are used to capture fuzzy linguistic evaluations in decision-making contexts. An FFN consists of two values: 'a,' the membership degree of the linguistic term in its designated term set (either performance or importance), and 'b,' the membership degree in the complementary set. For instance, a high performance rating, such as "Extremely High," corresponds to an FFN like (0.95, 0.15), meaning a strong membership in the "Extremely High" set and a lower membership in the complementary set. Conversely, a criterion such as "Fairly Unimportant" might have an FFN of (0.40, 0.70), signifying moderate membership in the "Fairly Unimportant" set and a higher membership in the complementary set.

In essence, this table offers a structured method for experts to linguistically evaluate alternative performances and criteria importance, integrating fuzzy logic to account for uncertainty and imprecision in assessments.

Step 3: Heterogeneity of Experts: Experts involved in the decision-making process possess diverse attributes, such as educational background (EQ), technical expertise (TE), managerial experience (ME), explicit knowledge (EK), and mental aptitude (MA). Each expert is evaluated based on these factors, and their importance weights are calculated. Table 2 is used to presents the ratings for these factors, providing a comprehensive view of how experts' characteristics influence their contributions to the decision-making process.

Table 2
Factors and ratings associated with experts' weights

Experts	Educational (w_{eq}=0.20)	Technical Experience (TE), (w_{te}=0.25)	Managerial (w_{me}=0.25)	Explicit (w_{ek}=0.20)	Mental (w_{ma} =0.10)
E_1	ME (2)	18	10	3	2
E_2	BE (1)	10	5	3	3
E_3	Ph. D.(3)	25	15	7	8
E_4	Ph. D. (3)	30	20	10	7

The table lists various factors that are considered in the decision-making process. These factors include: Educational Qualification (EQ), Technical Experience (TE) in years, Managerial Experience (ME), Explicit Knowledge (EK), and Mental Ability (MA). The table also presents the ratings assigned by different experts to each factor, along with their corresponding weights. The weights are denoted as follows: w_{eq}: Weight assigned to Educational Qualification; w_{te}: Weight assigned to Technical Experience; w_{me}: Weight assigned to Managerial Experience; w_{ek}: Weight assigned to Explicit Knowledge, and w_{ma}: Weight assigned to Mental Ability. These weights represent the importance or significance of each factor, as determined by the experts (E_1, E_2, E_3 and E_4). Under each factor, each expert has provided a rating or value. For example: E_1 has a Master's degree (ME) in Educational Qualification, 18 years of Technical Experience (TE), 10 years of Managerial Experience (ME), etc. E_2 has a Bachelor's degree (BE), 10 years of TE, 5 years of ME, etc. E_3 and E_4 both have Ph.D. degrees, with varying levels of experience in TE, ME, etc. Table 2 provides a structured overview of how experts' ratings and weights are associated with different factors. This information can be used to analyze and prioritize factors based on their importance as perceived by the experts.

Table 3 represents the intermediate assessment values and the weights of the experts on the basis of the above set of attributes. This table presents the evaluation of experts' weight coefficients for various criteria. Each row represents a different expert, and each column represents a specific criterion. Here's a breakdown of the table:

Table 3

Evaluation of experts' weight coefficients.

Experts	EQ	TE	ME	EK	MA	GM	Weight
E_1	0.13	0.15	0.13	0.06	0.03	0.06	0.17
E_2	0.07	0.08	0.06	0.06	0.04	0.05	0.12
E_3	0.20	0.21	0.19	0.15	0.10	0.12	0.33
E_4	0.20	0.25	0.25	0.18	0.09	0.14	0.38

EQ (Emotional Quotient), TE (Technical Expertise), ME (Management Expertise), EK (Experience Knowledge), MA (Market Awareness), and GM (General Management) are the criteria evaluated. Weight column represents the importance weight coefficients allotted to every criterion by the respective experts. For example, for Expert 1 (E_1): Emotional Quotient (EQ) has a weight coefficient of 0.13. Technical Expertise (TE) has a weight coefficient of 0.15. Management Expertise (ME) has a weight coefficient of 0.13. Experience Knowledge (EK) has a weight coefficient of 0.06. Market Awareness (MA) has a weight coefficient of 0.03. General Management (GM) has a weight coefficient of 0.06. Similarly, the table provides the weight coefficients assigned by each expert for all the criteria listed. These weight coefficients are essential for various decision-making processes, such as hiring, project management, or performance evaluation, where different criteria need to be weighted based on their importance or relevance to the task at hand.

Step 4: A matrix representing the criteria weights is formed by the DMs in terms of linguistic terms using the list of the ten degree linguistic terms as expressed in Table 4. It shows that the 4 experts estimate the weights of the criterion C_1 as VI, EI, SI and I respectively. It implies that expert E_2 awards the highest importance weight Extremely Important (EI) to the criterion market accessibility (C_1).

Table 4

Criteria weights in linguistic terms estimated by the expert group.

Experts	C_1	C_2	C_3	C_4	C_5
E_1	I	SI	VI	FI	I
E_2	FI	VI	EI	I	VI
E_3	EI	I	SI	SI	FI
E_4	VI	I	I	VI	I

In Table 4, we're presented with the linguistic criteria weights estimated by an expert group. Each criterion (C_1 to C_5) is evaluated by the experts (E_1 to E_4) and assigned linguistic weights representing their subjective assessment. Let's break it down:

Criteria (C_1 to C_5): These represent the different aspects or dimensions being evaluated. Without specific context, we can't determine what each criterion refers to, but they could represent various factors or considerations relevant to the subject under study.

Experts (E_1 to E_4): These are the individuals tasked with evaluating the criteria. Each expert has their own perspective and expertise, which influences their assessment of the criteria.

Linguistic Weights (I, SI, VI, FI, EI): Instead of numerical weights, linguistic terms are used to express the experts' judgments. Expert E_1 considers criterion C_2 as 'Insignificant', C_3 as 'Slightly Important', C4 as 'Very Important', and C_5 as 'Fairly Important', while no specific weight is assigned to C_1. Expert E_2 values C_1 as 'Fairly Important', C_2 as 'Very Important', C_3 as 'Extremely Important', and C_5 as 'Very Important', with no specific weight given to C_4. Expert E_3 ranks C_1 as 'Extremely Important', C_2 as 'Insignificant', C_3 as 'Slightly Important', and both C_4 and C_5 as 'Fairly Important'. Expert E_4 assigns 'Very Important' to C_1, 'Insignificant' to C_2, 'Insignificant' to C_3, 'Very Important' to C_4, and 'Insignificant' to C_5.

From Table 4, we can see variations in the weights assigned by different experts to different criteria. This reflects the subjective nature of expert judgment and highlights the importance of considering multiple perspectives when evaluating complex issues. Additionally, the use of linguistic terms allows for a more nuanced understanding of the experts' opinions compared to simple numerical weights.

Step 5: Linguistic weight are converted into the corresponding Fermatean fuzzy sets containing membership degree and non-membership degree, as expressed in Table 5. Fermatean fuzzy weights for the criteria are determined using the proposed method, which takes into account the preferences of the experts. Table 5 outlines the evaluations for theexperts (E_1 to E_4) on various criteria (C_1 to C_5) and provides insight into their weighted preferences.

Table 5
Fermatean fuzzy weights of the criteria by the proposed method based on expert preference.

Experts (weight)	C_1	C_2	C_3	C_4	C_5
E_1 (0.16)	(0.90, 0.20)	(0.60, 0.50)	(0.80, 0.30)	(0.70, 0.40)	(0.80, 0.30)
E_2(0.12)	(0.95, 0.15)	(0.90, 0.20)	(0.70, 0.40)	(0.80, 0.30)	(0.90, 0.20)
E_3 (0.33)	(0.60, 0.50)	(0.80, 0.30)	(0.95, 0.15)	(0.60, 0.50)	(0.70, 0.40)
E_4(0.38)	(0.80, 0.30)	(0.80, 0.30)	(0.90, 0.20)	(0.90, 0.20)	(0.80, 0.30)
AWDEW*	(0.76, 0.33)	(0.77, 0.32)	(0.86, 0.34)	(0.75, 0.34)	(0.77, 0.32)
Weight (w_j^d)	0.1585	0.1741	0.3505	0.1441	0.1728

*Average weights of criteria dominated by expert weight coefficients.

Each expert assigns a weight to each criterion, reflecting their perceived importance or preference for that criterion. These weights are expressed as Fermatean fuzzy numbers, which consist of two values: the membership degree and the non-membership degree. For example, Expert E_1 assigns a weight of (0.90, 0.20) to a particular criterion, where 0.90 indicates a high degree of membership, and 0.20 represents the non-membership degree for that criterion.

The Average Weighted Degree of Expert Weight (AWDEW) represents the average values of the membership and non-membership degrees across all criteria based on expert evaluations. The overall weight is derived by averaging the weights assigned by all experts, reflecting a combined judgment.

Expert Contributions: Expert E_4 has the largest influence on the overall weights (0.38), followed by E_3 (0.33), E_1 (0.16), and E_2 (0.12). This distribution shows that E_4 and E_3's judgments significantly affect the final weighting compared to E_1 and E_2.

Criteria Importance: The overall weights for the criteria reveal that C_3 is considered the most important (0.3505), while C_4 has the least weight (0.1441). This indicates that experts collectively prioritize C_3 more than the other criteria.

Fermatean Fuzzy Weights: By using Fermatean fuzzy numbers, this approach provides a more detailed representation of expert assessments, capturing both the degree of membership and the uncertainty inherent in expert judgments. This method enhances the accuracy of reflecting expert preferences in the decision-making process.

Table 5 illustrates the contributions of individual experts to the computation of the weights, emphasizing the relative importance of each criterion in the decision-making process.

Step 6: The Fermatean fuzzy aggregated weight matrix is derived by combining the individual Fermatean weight matrices, as specified in Eq.(4). Using the proposed approach, the aggregated weights for the criteria are computed as follows: (0.76, 0.329), (0.772, 0.317), (0.8675, 0.341), (0.748, 0.341), and (0.771, 0.318). These weights are then calculated according to the method outlined in Equation (5). The Fermatean fuzzy weighted average matrix, along with the corresponding expert weights, is presented in Table 5. The calculated importance weights for the criteria C_1, C_2, C_3, C_4, and C_5 are 0.1585, 0.1741, 0.3505, 0.1441, and 0.1728, respectively. This indicates that the expert group places the highest importance on criterion C_3 (market accessibility).

Step 7: The importance weight coefficient for each criterion is further determined through the application of the Fermatean fuzzy entropy weight method, which follows a three-step process. In the first step, a value is computed using Equation (6). The second step involves calculating the entropy ($\tilde{e}_j$) using Eq. (7). In the final step, the weight of criterion C_j is obtained using Equation (8). Based on this method, the weight values for criteria C_1, C_2, C_3, C_4, and C_5 are found to be 0.3265, 0.2188, 0.2649, 0.0838, and 0.1061, respectively. These weights, along with the intermediate assessment values, are displayed in Table 6.

Step 6: Fermatean fuzzy aggregated weight matrix is computed by aggregating the individual Fermatean weight matrices using Eq. (4). As per the proposed formula, the aggregated weights of the criteria are computed as (0.76, 0.329), (0.772, 0.317), (0.8675, 0.341), (0.748, 0.341) and (0.771, 0.318) respectively. The weights are calculated following Eq. (5), as per proposed weight measuring technique. Fermatean fuzzy weighted average matrix along with the weights of the experts using the proposed method is presented in Table 5. The importance weights for the criteria C_1, C_2, C_3, C_4, and C_5 are calculated as 0.1585, 0.1741, 0.3505, 0.1441, and 0.1728 respectively. It implies that the heterogeneous expert group provides the most importance weight to the criterion C_3 (market accessibility).

Step 7: The coefficient of importance weight of each criterion is also computed by the application of the Fermatean fuzzy entropy weight measuring method in three steps. In the first step, e_{ij} value is calculated using Eq.(6). In second step, entropy is ccalculated using Eq. (7). In the third step, weight of the criterion C_j is measured using Eq. (8). As per the Fermatean fuzzy entropy weighting method, the weights of the criteria C_1, C_2, C_3, C_4, and C_5 are determined as 0.3265, 0.2188, 0.2649, 0.0838 and 0.1061 respectively. The criteria weights and intermediate assessment values are presented in Table 6.

Table 6

Criteria weight measured by Fermatean fuzzy entropy method.

		C_1	C_2	C_3	C_4	C_5
e_{ij}	A_1	0.3675	0.6986	0.6638	0.9020	0.8961
	A_2	0.9729	0.9444	0.8199	0.9336	0.9256
	A_3	1.0097	0.9207	0.9864	0.9986	0.9665
	A_4	0.9978	0.9992	1.0008	0.9984	0.9999
$\overline{e}_{ij}$		0.8370	0.8907	0.8677	0.9582	0.9470
$1-\overline{e}_{ij}$		0.1630	0.1093	0.1323	0.0418	0.0530
Weight (w_j^e)		0.3265	0.2188	0.2649	0.0838	0.1061

Table 6 shows the computation of criteria weights using the Fermatean fuzzy entropy method. This approach is employed to assess the relative significance of the criteria (C_1 to C_5) by considering the evaluations provided by multiple decision-makers or alternatives (A_1 to A_4). The process of calculating the criteria weights is as follows:

- Each row corresponds to an alternative (A_1 to A_4), representing different perspectives, decision-makers, or scenarios.
- The values in each row indicate the relative importance or contribution of each criterion according to that particular alternative's assessment.
- For instance, for alternative A_1, the importance of criteria (C_1 to C_5) is evaluated as 0.3675, 0.6986, 0.6638, 0.9020, and 0.8961 respectively.

The row averages are calculated for the average importance of the criteria across all alternatives. This provides an overall perspective on the criteria's importance. The column averages indicate the average importance of each alternative across all criteria. This offers insight into the overall impact or preference of each alternative.

Overall Weights: The bottom row indicates the overall weight of each criterion, which is derived from the row averages. This reflects the combined assessment of all alternatives and represents the final determination of criteria importance. In this case, the weights for criteria C_1 to C_5 are found as 0.3265, 0.2188, 0.2649, 0.0838, and 0.1061 correspondingly.

Relative Importance of Criteria: Criterion C_1 holds the highest weight (0.3265), followed by C_3 (0.2649), C_2 (0.2188), C_5 (0.1061), and C_4 (0.0838). This implies that, collectively, the decision-makers or alternatives perceive C_1 as the most important criterion and C_4 as the least important.

Assessment Consistency: The consistency of assessments across different alternatives can be inferred by comparing their evaluations. Consistency suggests a higher level of agreement among decision-makers regarding the importance of criteria.

Methodology Significance: The application of the Fermatean fuzzy entropy method allows for the inclusion of and ambiguity uncertainty in the process, leading to a more comprehensive evaluation of the importance of various criteria. Table 6 offers valuable insights into the relative significance of criteria from multiple decision-maker perspectives, thereby supporting more informed decision-making.

Step 8: The coefficient representing the decision-making attitude, based on the preferences of the decision-makers, is taken into account. The combined weights for the criteria are then calculated using Eq.(9) and presented in Table 7. The resulting combined weights for criteria C_1, C_2, C_3, C_4, and C_5 are 0.2148, 0.2218, 0.2602, 0.1824, and 0.1209, in that order. This indicates that the highest importance weight has been assigned to criterion C_3 (Market accessibility).

Table 7
Combined weight of each criterion* .

Weight	C_1	C_2	C_3	C_4	C_5
Decision makers preference (w_j^d)	0.1585	0.1741	0.3506	0.1441	0.1728
Entropy weighting method (w_j^e)	0.2711	0.2694	0.1698	0.2207	0.0690
Combined weight (w_j)	0.2148	0.2218	0.2602	0.1824	0.1209

*Value of ξ has been taken as 0.5.

Step 9: Each heterogeneous experts forms a decision matrix as per the Eq. (10). It consists of linguistic performance ratings of the alternatives warehouse locations. Table 8 is dedicated for presenting the decision matrices. The problem involves selecting an optimal warehouse location from four alternatives (A_1, A_2, A_3, A_4) based on five criteria (C_1 to C_5) evaluated by four experts (E_1 to E_4). Each criterion is assessed using linguistic variables such as Very Low (VL), Low (L), High (H) and Very High (VH), with the first four criteria being positively oriented (higher values preferred) and the last criterion (C_5) negatively oriented (lower values preferred). The matrix reveals varying preferences among the experts, highlighting that while some alternatives (particularly A_1) are favored across multiple criteria, others (such as A_4) excel in specific areas. The diversity in expert evaluations suggests the need for a balanced approach in decision-making, possibly incorporating weighted factors or pairwise comparisons to refine the selection process.

Table 8
Decision Matrices Represented by Linguistic Variables Estimated by Experts for Warehouse Location Selection

Experts	Alternatives	C_1(+)	C_2(+)	C_3(+)	C_4(+)	C_5(-)
E_1	A_1	VH	SH	EH	H	L
	A_2	H	L	H	L	VL
	A_3	SH	EH	H	L	H
	A_4	EH	H	L	VH	SH
E_2	A_1	VH	FH	EH	H	L
	A_2	FH	SH	EH	VH	H
	A_3	L	VH	SH	H	H
	A_4	VL	H	L	H	L
E_3	A_1	EH	VH	H	L	H
	A_2	SH	EH	VH	H	EH
	A_3	H	L	VH	SH	H
	A_4	VH	H	L	H	H
E_4	A_1	EH	VH	H	EH	VH
	A_2	SH	EH	VL	H	L
	A_3	L	VH	SH	H	H
	A_4	FH	SL	EH	VH	H

Step 10: The linguistic variables from each decision matrix are transformed into Fermatean fuzzy (FF) sets, which include both membership and non-membership grades, as defined by Equation (11). Table 9 presents the resulting Fermatean fuzzy decision matrices. For instance, the linguistic rating "VH" (very high), assigned by expert E_1 to alternative A_1 (Haldia), is converted into the Fermatean fuzzy number (0.90, 0.20). Here, 0.90 represents the degree of membership, and 0.20 represents the degree of non-membership.

Table 9
Fermatean Fuzzy Decision Matrices

Experts	**Alternatives**	**C_1 (+)**	**C_2 (+)**	**C_3 (+)**	**C_4 (+)**	**C_5 (-)**
E_1	A_1	(0.900, 0.200)	(0.600, 0.500)	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)
	A_2	(0.800, 0.300)	(0.500, 0.600)	(0.800, 0.300)	(0.300, 0.800)	(0.200, 0.900)
	A_3	(0.600, 0.500)	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)
	A_4	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)
E_2	A_1	(0.900, 0.200)	(0.700, 0.400)	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)
	A_2	(0.700, 0.400)	(0.600, 0.500)	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)
	A_3	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.800, 0.300)	(0.800, 0.300)
	A_4	(0.200, 0.900)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)	(0.200, 0.900)
E_3	A_1	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)
	A_2	(0.600, 0.500)	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.950, 0.150)
	A_3	(0.800, 0.300)	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.800, 0.300)
	A_4	(0.900, 0.200)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)	(0.800, 0.300)
E_4	A_1	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.905, 0.105)	(0.900, 0.200)
	A_2	(0.600, 0.500)	(0.950, 0.150)	(0.200, 0.900)	(0.800, 0.300)	(0.150, 0.950)
	A_3	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.800, 0.300)	(0.800, 0.300)
	A_4	(0.700, 0.400)	(0.500, 0.600)	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)

Step 11: In this case, a cost category criterion is applied. As a result, the normalization of the Fermatean fuzzy ratings in the decision matrix is performed using Eq. (12). The normalized Fermatean fuzzy decision matrices are shown in Table 10. For instance, the Fermatean fuzzy performance rating (0.300, 0.800) for the cost criterion is normalized to (0.800, 0.300).

Table 10

Normalized Fermatean fuzzy decision matrices

Experts	Alternatives	C_1	C_2	C_3	C_4	C_5
E_1	A_1	(0.900, 0.200)	(0.600, 0.500)	(0.950, 0.150)	(0.800, 0.300)	(0.800, 0.300)
	A_2	(0.800, 0.300)	(0.500, 0.600)	(0.800, 0.300)	(0.300, 0.800)	(0.900, 0.200)
	A_3	(0.600, 0.500)	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)	(0.300, 0.800)
	A_4	(0.950, 0.150)	(0.800, 0.300)	(0.300, 0.800)	(0.900, 0.200)	(0.500, 0.600)
E_2	A_1	(0.900, 0.200)	(0.700, 0.400)	(0.950, 0.105)	(0.800, 0.300)	(0.800, 0.300)
	A_2	(0.700, 0.400)	(0.600, 0.500)	(0.950, 0.105)	(0.900, 0.200)	(0.300, 0.800)
	A_3	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.800, 0.300)	(0.300, 0.800)
	A_4	(0.200, 0.900)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)	(0.900, 0.200)
E_3	A_1	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.300, 0.800)	(0.300, 0.800)
	A_2	(0.600, 0.500)	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.150, 0.950)
	A_3	(0.800, 0.300)	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.300, 0.800)
	A_4	(0.900, 0.200)	(0.800, 0.300)	(0.300, 0.800)	(0.800, 0.300)	(0.300, 0.800)
E_4	A_1	(0.950, 0.150)	(0.900, 0.200)	(0.800, 0.300)	(0.950, 0.150)	(0.200, 0.900)
	A_2	(0.600, 0.500)	(0.950, 0.150)	(0.200, 0.900)	(0.800, 0.300)	(0.950, 0.150)
	A_3	(0.300, 0.800)	(0.900, 0.200)	(0.600, 0.500)	(0.800, 0.300)	(0.300, 0.800)
	A_4	(0.700, 0.400)	(0.500, 0.600)	(0.950, 0.150)	(0.900, 0.200)	(0.300, 0.800)

Step 12: There are four experts in given example. Each expert forms a decision matrix. Therefore, therefore are four such decision matrices. These decision matrices are firstly weighted by the respective weight of the each expert and then are aggregated using Eq. (13). The aggregated weighted normalised decision matrix (AWNDM) has been shown in Table 11.

Table 11

AWNDM in Fermatean fuzzy numbers.

Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	(0.926, 0.162)	(0.819, 0.270)	(0.834, 0.255)	(0.684, 0.405)	(0.690, 0.399)
A_2	(0.555, 0.358)	(0.619, 0.294)	(0.751, 0.161)	(0.637, 0.276)	(0.649, 0.264)
A_3	(0.312, 0.469)	(0.655, 0.125)	(0.492, 0.289)	(0.403, 0.378)	(0.568, 0.213)
A_4	(0.361, 0.057)	(0.304, 0.114)	(0.114, 0.304)	(0.342, 0.076)	(0.228, 0.190)

The Fermatean fuzzy weighted average (FFWA) operator is employed to combine matrices created by each expert. The FFWA performance score of alternative A_1 concerning criterion C_1 is computed as follows.

$$\sum_{k=1}^{4} w_{ek} * \alpha_{11}^{kN}$$
$$= (0.17\times0.90 + 0.12\times0.90 + 0.33\times0.95 + 0.38\times0.95,\ 0.17\times0.20 + 0.12\times0.20 + 0.33\times0.15 + 0.38\times0.15)$$
$$= (0.926,\ 0.162)$$

Step 13: FFPIS and FFNIS are determined based on score function. FFPISs (I^+) are determined by using Eq. (14) FFNISs (I^+) are determined by the application of the Eq. (15). The FFPIs and FFNIS are shown in Table 12. I^+ and I^- are presented below.

I^+ = (0.926, 0.162), (0.819, 0.270), (0.834, 0.255), (0.684, 0.405), (0.690, 0.399).

I^- = (0.312, 0.469), (0.304, 0.114), (0.114, 0.304), (0.403, 0.378), (0.228, 0.190).

Table 12

FFPIS (I^+) and FFNIS (I^-).

Alternatives	C_1	C_2	C_3	C_4	C_5
FFPIS (I^+)	(0.926, 0.162)	(0.819, 0.270)	(0.834, 0.255)	(0.684, 0.405)	(0.690, 0.399)
FFNIS (I^-)	(0.312, 0.469)	(0.304, 0.114)	(0.114, 0.304)	(0.403, 0.378)	(0.228, 0.190)

Step 14: Determination of the weighted Euclidean distances between alternatives and the FPIS and FNIS. The weighted Euclidean distances between alternatives and the FFPIS are measured by the Eq. (16) as follows. The Euclidean distances of the alternative A_2 from the FFPISs are 0.704, 0.457, 0.448, 0.021 and 0.348 respectively. The combined weights have already been determined as 0.2148, 0.2218, 0.2602, 0.1824 and 0.1209 respectively. Therefore, the weighted Euclidean distances between A2 and the FPIS is measured as follows:

$$D\left(A_2, I^+\right) = 0.704 \times 0.2148 + 0.457 \times 0.2218 + 0.448 \times 0.2602 + 0.021 \times 0.1824 + 0.348 \times 0.1209 = 0.4149.$$

Similarly, weighted Euclidean distances of the remaining alternatives are computed and found as 0, 0.4149, 0.5161 and 0.4317 respectively.

The Euclidean distances of the alternative A_2 from the FFNISs are 0.7046, 0.8648, 0.8601, 0.7023 and 0.4781 respectively. The weighted Euclidean distance (WED) between alternative A_2 and the FNIS is measured as follows.

$$D\left(A_2, I^-\right) = 0.704 \times 0.7046 + 0.457 \times 0.8648 + 0.448 \times 0.8601 + 0.021 \times 0.7023 + 0.348 \times 0.4781 = 0.7528.$$

Correspondingly, WED of the other alternatives from the negative ideal solutions are computed as 0.6729, 0.7528, 0.7358 and 0.6597 in that order.

The smaller value of weighted distance guarantees the superior alternative. Smaller value specifies that the separation of the alternative is nearer to the FFPIS. The weighted Euclidean distances between alternatives and the FFNIS, is measured by the Eq. (17) and have been shown in Table 13.

Table 13
Relative weighted distance measure, proposed proximity coefficient and ranking order of alternatives.

Alternatives	$\frac{D(A_i, I^-)}{D_{\max}(A_i, I^-) + D_{\min}(A_i, I^-)}$	$\frac{D(A_i, I^+)}{D_{\max}(A_i, I^+) + D_{\min}(A_i, I^+)}$	$PC(A_i)$	Rank
A_1	0 .4764	0 .0000	0 .4764	1
A_2	0 .5330	0 .8040	-0 .2710	2
A_3	0 .5209	1 .0000	-0 .4791	4
A_4	0 .4670	0 .8365	-0 .3695	3

Step 15: *Proximity coefficient and ranking:* Proximity coefficient of alternative is measured using Eq. (18)**.** It is observed that warehouse location (A1) that is Haldia has attained the highest proximity coefficient 0.4764. The calculation procedure of proximity coefficient is demonstrated below.

Here, $D(A_1, I^-) = 0.6729,$ $D(A_1, I^+) = 0$, $D_{\max}(A_i, I^-) = 0.7528,$ and $D_{\min}(A_i, I^-) = 0.6597,$ so we have

$$PC\left(A_1\right) = \frac{0.6729}{0.7528 + 0.6597} - \frac{0}{0.7528 + 0.6597} = 0.4764.$$

Similarly, the proximity coefficients of all the alternatives are measured as 0.4764, − 0.2710, − 0.4791, and −0.3695 respectively.

Warehouse location Durgapur (A_2) has attained proximity coefficient -0.2710, second highest. Warehouse location Barrackpore (A_4) has obtained -0.3695 as the proximity coefficient, third highest. Lastly, Asansol has obtained the least proximity coefficient -0.479. Thus, Haldia is ranked 1st, Durgapur 2nd, Bardhaman 3rd, and Asansol 4th. Therefore the ranking order is:

$$A_1 > A_2 > A_4 > A_3.$$

Table 13 provides a detailed overview of the relative weighted distance measure, the proposed proximity coefficient, and the corresponding ranking order of the alternatives. It presents the calculated values for each alternative based on the defined criteria, followed by their proximity coefficients, which indicate how closely they align with the ideal solution. The ranking order reflects the relative performance of the alternatives, with the top-ranked alternative demonstrating the best overall fit according to the weighted distance and proximity measures.

Fig.4 depicts the proximity coeffiients of the alternatives by the proposed method. Proposed Proximity Coefficient is a measure proposed to assess the proximity or closeness of alternatives to a certain reference point or ideal solution. It involves calculations or coefficients that quantify how closely each alternative aligns with the desired criteria.

Fig. 5 visually illustrates the ranking orders of the warehouse locations based on the proposed proximity coefficients. The diagram effectively displays how each warehouse location is ranked relative to others, with the proximity coefficients used as a measure of their closeness to the ideal solution. This graphical representation helps to easily compare the performance of the locations, providing a clear view of their relative standings in terms of the defined criteria. The ranking is presented in a manner that highlights the optimal choices for warehouse placement, aiding in decision-making.

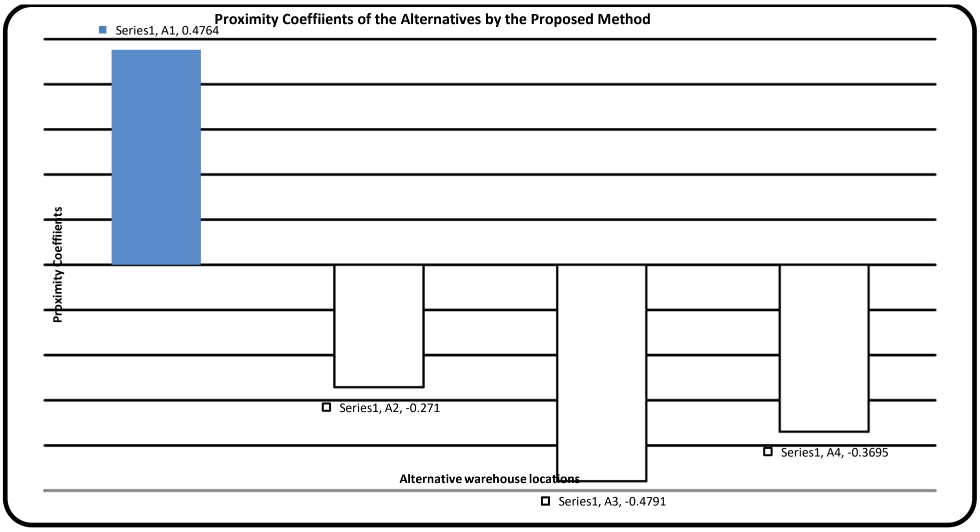

Fig. 4. Proximity coeffiients of the alternatives by the proposed method.

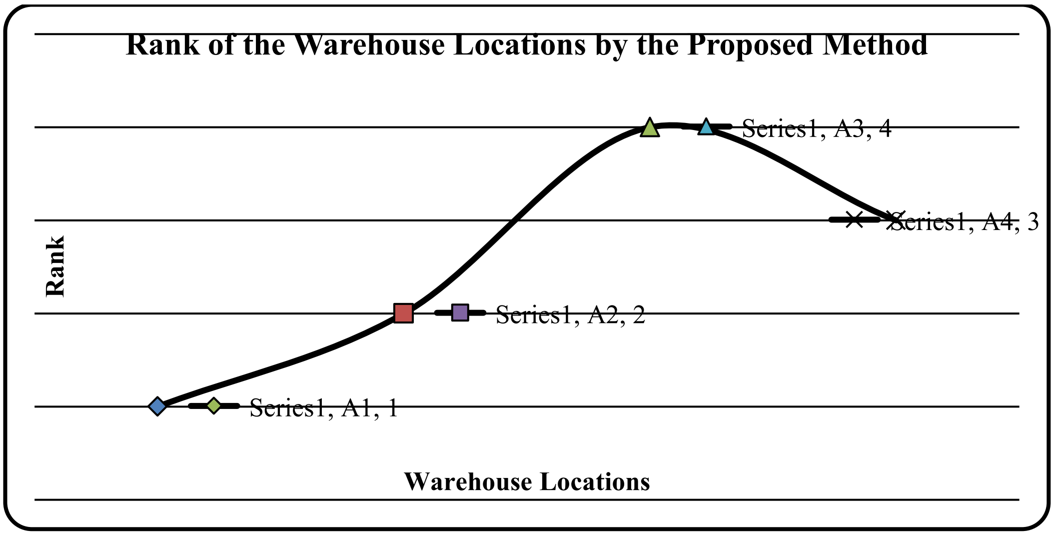

Fig. 5. Rank of the Warehouse Locations by the Proposed Method.

Step 16: *Closeness coefficient and ranking:* Closeness coefficient of an alternative is computed using the Eq. (19). It is found that closeness coefficients of the warehouse alternatives Haldia, Durgapur, Aansol and Bardhaman are 1.0000, 0.6447, 0.5877 and 0.6044 respectively. Therefore, the warehouse locations Haldia, Durgapur, Asansol and Bardhaman have been ranked 1st, 2nd, 4th and 3rd respectively. Thus, the ranking order is $A_1>A_2>A_4>A_3$. Table 14 shows weighted distance measures, relative closeness and rank of the alternatives.

Table 14
Weighted distance measures, Relative Closeness and rank of the alternatives .

Alternatives	$D(A_i, I^-)$	$D(A_i, I^+)$	$RC(A_i)$	Rank
A_1	0 .6729	0 .0000	1 .0000	1
A_2	0 .7528	0 .4149	0 .6447	2
A_3	0 .7358	0 .5161	0 .5877	4
A_4	0 .6597	0 .4317	0 .6044	3

It is easily seen that the ranks obtained by existing and proposed method are absolutely identical. Fig.6 depicts the closeness coefficient of the warehouse locations under consideration.

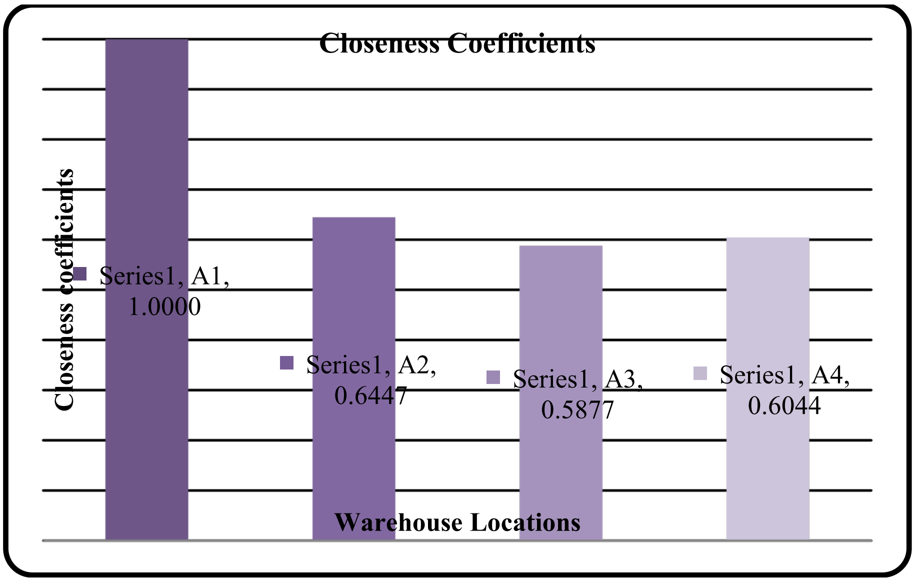

Fig.6. Closeness coefficient of the locations under consideration.

Fig.7 graphically presents the ranking orders of the warehouse locations as per closeness coefficients.

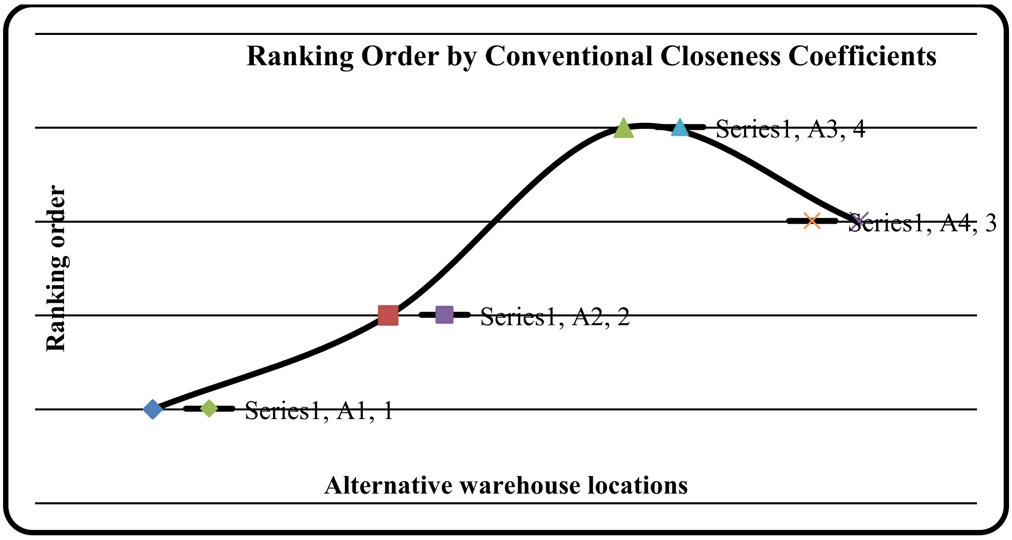

Fig.7. Ranking orders of the warehouse locations by closeness coefficients.

Step 17: *Combined selection index (CSI) and ranking:* In this section of the paper, we propose the combined selection index (CSI), comprising of proximity index and closeness coefficient. Here we assign $\lambda = 0.5$. The combined selection indices of the warehouse alternatives viz. Haldia, Durgapur, Aansol and Bardhaman have been determined as 0.7382, 0.1869, 0.0543 and 0.1175 respectively. The CSIs are computed by using Eq. (20). The four locations of warehouses viz. Haldia, Durgapur, Asansol and Bardhaman are ranked as 1, 2, 4 and 3 in that order. Hence, the ranking order is $A_1>A_2>A_4>A_3$.

Table 15
Linear combination of coefficient of proximity with relative closeness and the alternatives' rank.

Alternatives	$CP(A_i)$	$RC(A_i)$	$RC(A_i)$	Rank
A_1	0.4764	1.0000	0 .7382	1
A_2	-0 .2710	0 .6447	0 .1869	2
A_3	-0 .4791	0 .5877	0 .0543	4
A_4	-0 .3695	0 .6044	0 .1175	3

Table 15 presents the results of the linear combination of proximity coefficients, relative closeness coefficients, and the ranking order of four alternative warehouse locations. The table lists each alternative (A_1, A_2, A_3, and A_4), along with its proximity coefficient, relative closeness, and combined selection index (CSI). The combined CSI ranks the alternatives based on their respective proximity and closeness coefficient. A_1 ranks the highest with a CSI of 0.7382, followed by A_2 (0.1869), A_4 (0.1175), and A_3 (0.0543), which ranks the last. Notably, the ranks derived from the proximity and closeness coefficients match those obtained using the combined CSI, suggesting that the combination method does not alter the ordering of the alternatives. Fig. 8 visually illustrates the CSIs, reinforcing the consistency in ranking.

Fig.8. Combined selection indices for the alternative warehouse locations.

Fig. 9 diagrammatically represents the ranking orders of the alternative warehouse locations based on the combined selection indices. This visual representation provides a clear comparison of the warehouse locations by displaying their rankings according to the integrated selection criteria. The combined selection indices take into account multiple factors, giving a holistic view of each location's suitability. By using this approach, the diagram highlights the relative strengths of each alternative, enabling a more informed decision-making process for selecting the most optimal warehouse location.

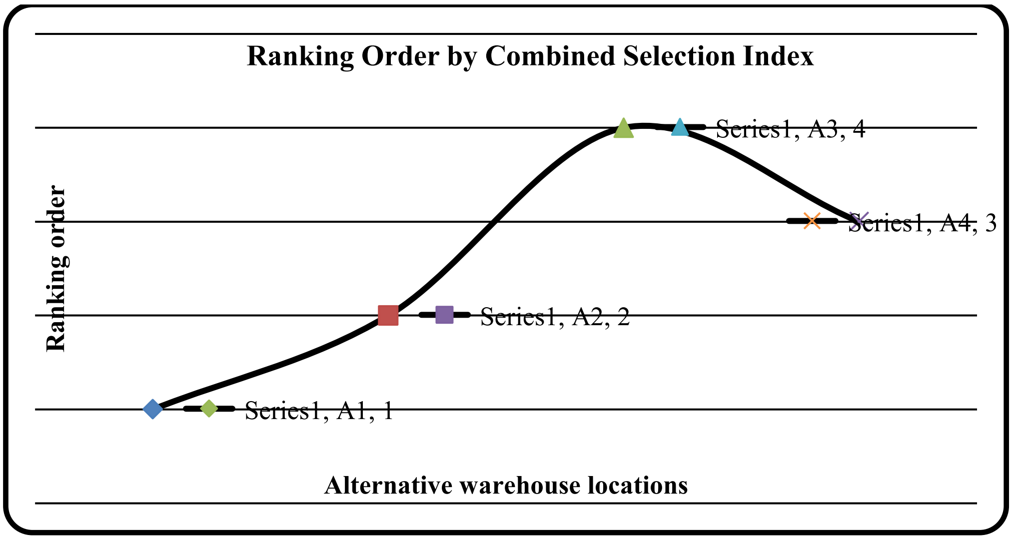

Fig.9. Ranking otders of the alternative warehouse locations by combined selection indices.

5. Sensitivity Analysis

This section presents a sensitivity analysis conducted to evaluate the robustness of the developed and applied method. Firstly, sensitivity is conducted with the variation of coefficient of decision makers' preference (ξ) for the weights measured by the proposed method and Fermatean fuzzy entropy weighting method. Secondly, it is carried out with the variation of parameter λ, which is the coefficient of proximity index obtained by the proposed method. In both the cases, the effect of the changes of these parameters on selection indices, ranking order and the decision have been observed and analyzed.

5.1 Sensitivity analysis with respect variation of ξ

Table 16 shows the proximity closeness of the alternative warehouse locations for different values of ξ. Sure, here's how you can illustrate Table 16. This representation aligns the data in a tabular format, with each row representing an alternative warehouse location and each column representing a different value of ξ. The numerical values are listed under each corresponding ξ value for each alternative.

Table 16

Proximity of the warehouse locations for different values of ξ.

Alternative	ξ=0	ξ=0.1	ξ=0.2	ξ=0.3	ξ=0.4	ξ=0.5	ξ=0.6	ξ=0.7	ξ=0.8	ξ=0.9	ξ=1.0
A_1	0 .4764	0 .4760	0 .4756	0 .4752	0 .4747	0 .4743	0. 4739	0.4735	0 .4731	0.4727	0. 4723
A_2	-0 .2710	-0. 2740	-0 .2769	-0 .2799	-0 .2829	-0 .2859	- 0 .2890	-0.2921	- 0 .2952	-0 .2983	-0. 3015
A_3	-0 .4791	-0. 4786	-0 .4781	-0 .4777	-0 .4772	-0 4767	- 0 .4762	-0 .4758	- 0 .4753	-0 .4748	-0. 4743
A_4	-0 .3695	-0 .3734	-0 .3774	-0 .3814	-0 .3854	-0. 3895	- 0 .3936	- 0 .3977	- 0 .4019	-0 .4061	-0. 4104

Table 17 presents the closeness coefficients (CCs) of the alternative warehouse locations for different values of ξ.

Table 17

Closeness coefficients (CC) of the alternative warehouse locations for varying ξ.

Alternative	ξ =0	ξ =0 .10	ξ =0 .20	ξ =0 .3	ξ =0 .4	ξ = 0 .5	ξ = 0 .6	ξ = 0 .7	ξ =0 .8	ξ = 0 .9	ξ= 1 .0
A_1	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000	1 .0000
A_2	0 .6447	0 .6447	0 .6447	0 .6448	0 .6448	0 .6449	0 .6449	0 .6449	0 .6450	0 .6450	0 .6451
A_3	0 .5877	0 .5889	0 .5900	0 .5912	0 .5923	0 .5935	0 .5946	0.5958	0 .5970	0 .5981	0 .5993
A_4	0 .6044	0 .6042	0 .6040	0 .6038	0 .6036	0 .6034	0 .6032	0 .6030	0 .6027	0 .6025	0 .6023

The corresponding combined selection indices (CSIs) have been measured of the alternative warehouse locations for different those values of ξ and have been presented in Table 18.

Table 18

Combined selection index (CSI) of the warehouse locations for varying ξ.

Alternative	ξ =0	ξ =0 .10	ξ =0 .20	ξ =0 .3	ξ =0 .4	ξ = 0 .5	ξ = 0 .6	ξ = 0 .7	ξ =0 .8	ξ = 0 .9	ξ= 1 .0
A_1	0 .7382	0 .7380	0 .7378	0 .7376	0 .7374	0 .7372	0 .7370	0 .7368	0 .7366	0 .7363	0 .7361
A_2	0 .1868	0 .1854	0 .1839	0 .1824	0 .1810	0 .1795	0 .1780	0 .1764	0 .1749	0 .1733	0 .1718
A_3	0 .0543	0 .0551	0 .0559	0 .0567	0 .0576	0 .0584	0 .0592	0 .0600	0 .0608	0 .0617	0 .0625
A_4	0 .1175	0 .1154	0 .1133	0 .1112	0 .1091	0 .1070	0 .1048	0 .1026	0 .1004	0 .0982	0 .0960

The value of the parameter has been varied from ξ=0 to ξ=1.0 with a increment of 0.1. The values in Table 18 represent the Combined Selection Index (CSI) of warehouse locations for different values of ξ. The CSI is a composite measure used to evaluate and compare the suitability of alternative warehouse locations based on multiple criteria or attributes. Here's the significance of the values in the table: The CSI values for Alternative A_1 range from 0.7361 to 0.7382 across different values of ξ. Higher CSI values indicate greater suitability or desirability of the warehouse location. A decreasing trend in CSI values with increasing ξ suggests a decreasing level of suitability as the parameter ξ increases. The CSI values for Alternative A_2 range from 0.1718 to 0.1868 across different values of ξ. A^1 though lower than those of A_1, these values still reflect a considerable level of suitability. Similar to A_1, a decreasing trend in CSI values with increasing ξ indicates decreasing suitability. The CSI values for Alternative A3 range from 0.0543 to 0.0625 across different values of ξ. These values are notably lower compared to A^1 and A2, suggesting a lower level of suitability for this warehouse location. The increasing trend in CSI values with increasing ξ suggests a slight improvement in suitability as ξ increases. The CSI values for Alternative A_4 range from 0.0960 to 0.1175 across different values of ξ. These values are lower compared to A_1 and A_2 but higher than those of A_3, indicating a moderate level of suitability. Similar to A_3, the increasing trend in CSI values with increasing ξ suggests a slight improvement in suitability as ξ increases. The CSI values provide insights into the relative suitability of alternative warehouse locations under varying conditions represented by different values of ξ. The analysis of these values can inform decision-makers in selecting the most appropriate warehouse location based on their specific requirements and preferences.

Table 19

Ranking orders of the warehouse locations by Proximity Closeness (PC) for varying values of ξ.

Alternatives	ξ=0	ξ=0.1	ξ=0.2	ξ=0.3	ξ=0.4	ξ=0.5	ξ=0.6	ξ=0.7	ξ=0.8	ξ=0.9	ξ=1.0
A_1	1	1	1	1	1	1	1	1	1	1	1
A_2	2	2	2	2	2	2	2	2	2	2	2
A_3	4	4	4	4	4	4	4	4	4	4	4
A_4	3	3	3	3	3	3	3	3	3	3	3

Table 19 shows the respective ranking order of the alternative warehouse locations according to their proximity coefficients.

Table 20

Ranking orders of the warehouse locations by Closeness coefficients (CC) for different values of ξ.

Alternatives	ξ=0	ξ=0.1	ξ=0.2	ξ=0.3	ξ=0.4	ξ=0.5	ξ=0.6	ξ=0.7	ξ=0.8	ξ=0.9	ξ=1.0
A_1	1	1	1	1	1	1	1	1	1	1	1
A_2	2	2	2	2	2	2	2	2	2	2	2
A_3	4	4	4	4	4	4	4	4	4	4	4
A_4	3	3	3	3	3	3	3	3	3	3	3

Table 20 shows the ranking order of the alternative warehouse locations according to their closeness coefficients**. Table 21** presents the relevant ranking order of the alternative warehouse locations as per their combined selection indices**.** It is observed that there is no change in the ranking order and decision regarding the selection of the warehouse location alternatives, though combined selection indices vary incrementally. The ranking orders for the locations, Haldia, Durgapur, Asansol and Bardhaman are 1, 2, 4, and 3 respectively for all values of ξ. Therefore, it clearly shows that the proposed method is absolutely robust with respect to the variation of ξ.

Table 21

Ranking orders of the warehouse locations by combined selection indices for varying ξ.

Alternatives	ξ=0	ξ=0.1	ξ=0.2	ξ=0.3	ξ=0.4	ξ=0.5	ξ=0.6	ξ=0.7	ξ=0.8	ξ=0.9	ξ=1.0
A_1	1	1	1	1	1	1	1	1	1	1	1
A_2	2	2	2	2	2	2	2	2	2	2	2
A_3	4	4	4	4	4	4	4	4	4	4	4
A_4	3	3	3	3	3	3	3	3	3	3	3

Fig.10 is the graphiical representation of the combined weights under discretely varying ξ.

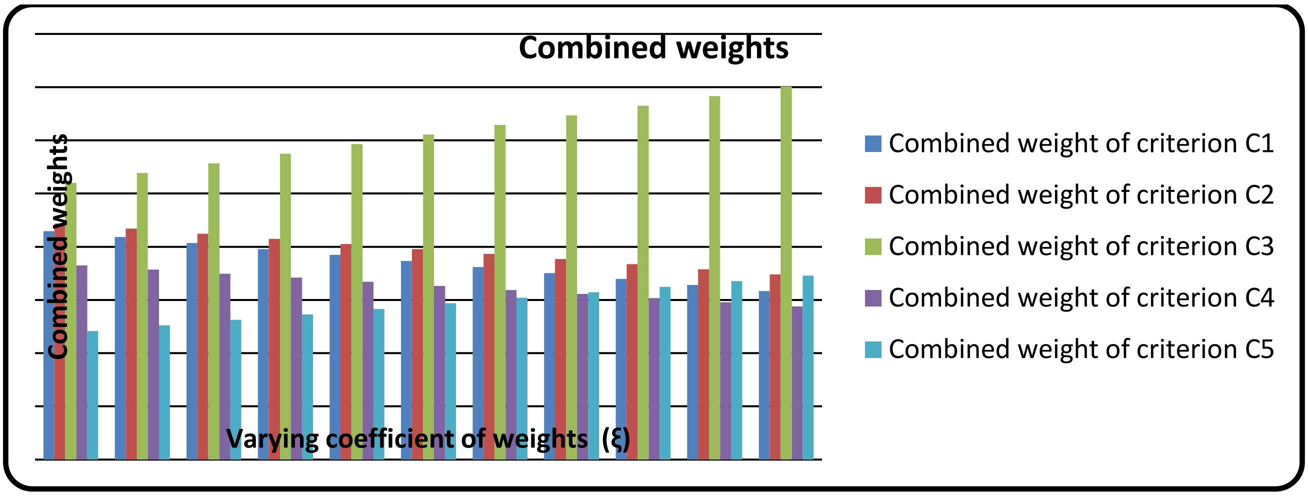

Fig.10. Combined weights under discretely varying ξ

Fig.11 diagramatically shows the proximity coefficients of the alternative warehouse locations for different selected values of ξ.

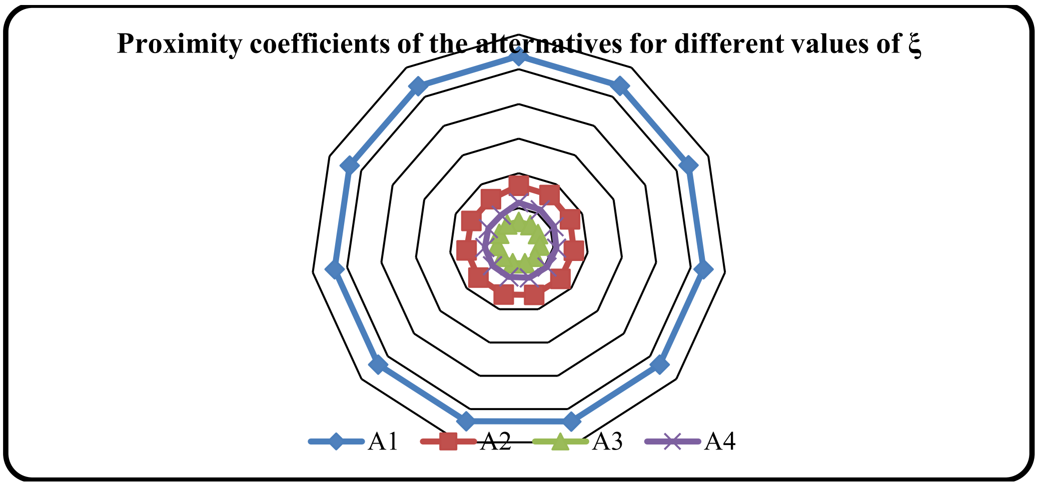

Fig.11. Proximity oefficients of the alternative warehouse locations for different values of ξ.

Fig.12 depicts the closeness oefficients of the alternative warehouse locations for diverse values of the parameter ξ.

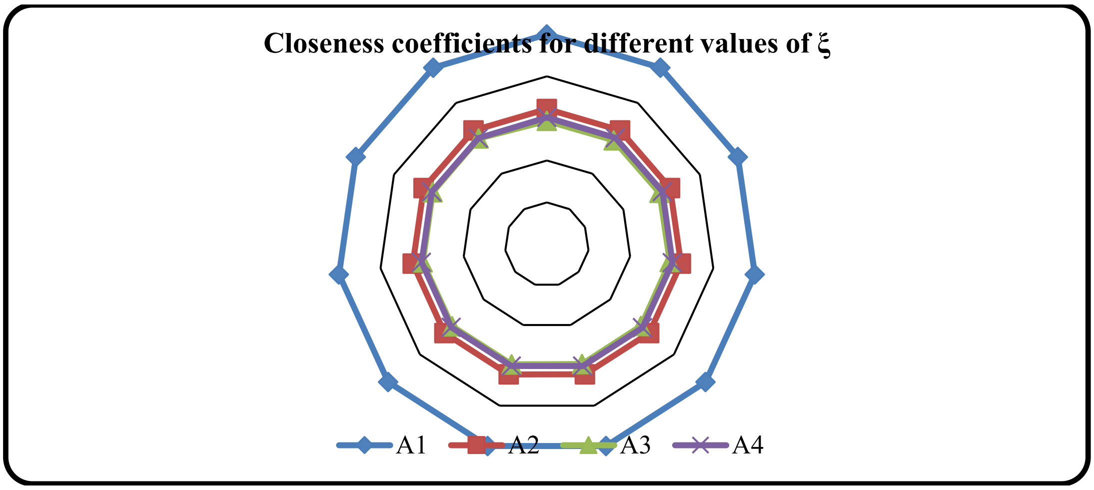

Fig.12. Closeness oefficients of the alternative warehouse locations for different values of ξ.

Fig.13 portraits the combined selection indices of the alternatives for different values of ξ.

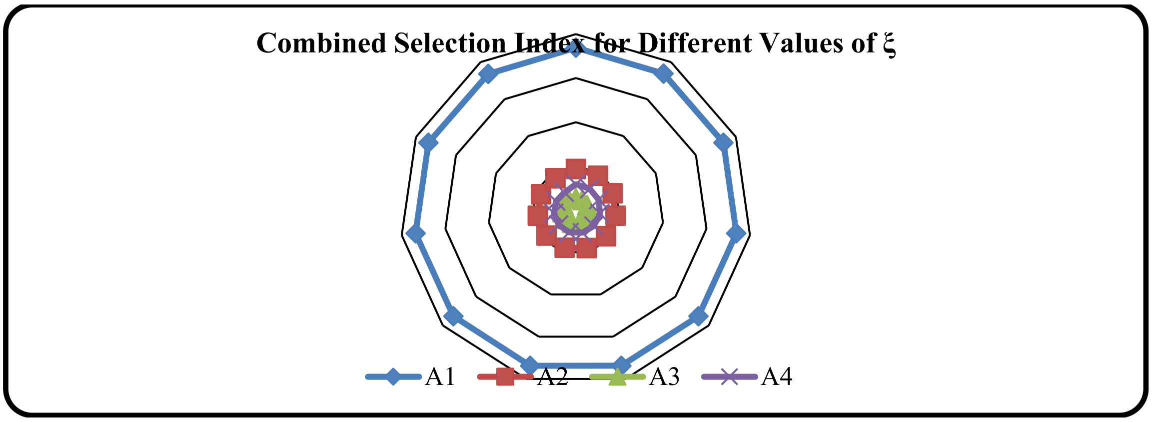

Fig.13. Combined selection indies of the alternative warehouse locations for different values of ξ.

Fig.14 diagramatically compares the ranks of the warehouse locations for diverse incremental values of ξ.

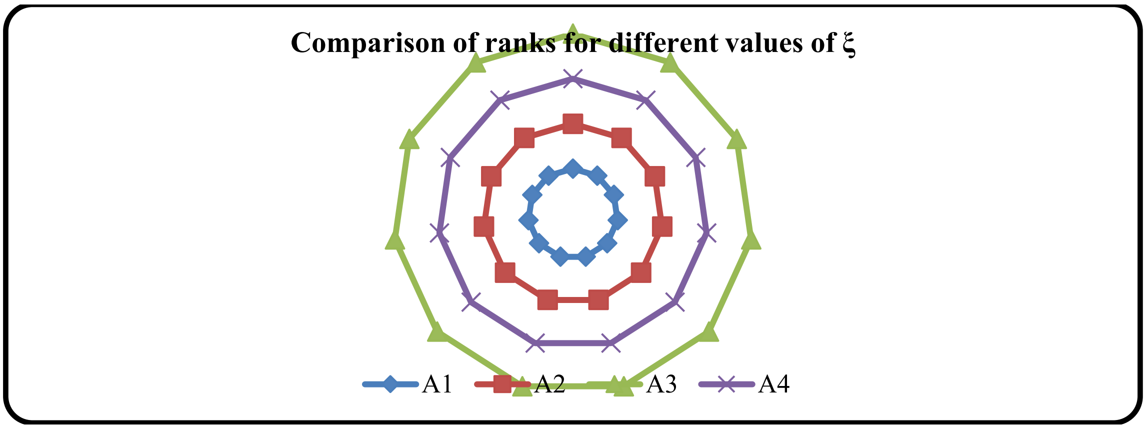

Fig.14. Comparison of ranks for different values of ξ.

5.2 Sensitivity analysis with respect to variation of the parameter λ

Sensitivity analysis of combined selection index (CSI), based on varying values of the parameter λ , has been conducted and the associated values of CSI with corresponding ranking orders have also been presented in Table 22.

Table 22

Sensitivity analysis of Combined Selection Index (CSI) based on varying λ .

	Combined Selection Index $CSI(A_i)$				Rank of the alternative warehouse locations			
λ Value	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
$\lambda = 0$	1.0000	0.6447	0.5877	0.6044	1	2	4	3
$\lambda = 0.1$	0.9476	0.5531	0.4810	0.5070	1	2	4	3
$\lambda = 0.2$	0.8953	0.4616	0.3743	0.4096	1	2	4	3
$\lambda = 0.3$	0.8429	0.3700	0.2677	0.3122	1	2	4	3
$\lambda = 0.4$	0.7906	0.2784	0.1610	0.2148	1	2	4	3
$\lambda = 0.5$	0.7382	0.1869	0.0543	0.1175	1	2	4	3
$\lambda = 0.6$	0.6858	0.0953	-0.0524	0.0201	1	2	4	3
$\lambda = 0.7$	0.6335	0.0037	-0.1591	-0.0773	1	2	4	3
$\lambda = 0.8$	0.5811	-0.0879	-0.2657	-0.1747	1	2	4	3
$\lambda = 0.9$	0.5288	-0.1794	-0.3724	-0.2721	1	2	4	3
$\lambda = 1.0$	0.4764	-0.2710	-0.4791	-0.3695	1	2	4	3

It shows that each alternative warehouse location bears a constant ranking order without any change for all selected values of λ. The ranking orders for the warehouse locations, Haldia, Durgapur, Asansol and Bardhaman are 1st, 2nd, 4th, and 3rd respectively for every value of λ. In other word, the ranking order is $A_1>A_2>A_4>A_3$. This also ensures that the ranks of the WLs are alike for sensitivity analysis with respect to changes of λ and ξ . Therefore, it obviously confirms that the proposed method is totally robust with respect to varying λ. Fig.15 graphically represents the combined selection indices for discreate λ.

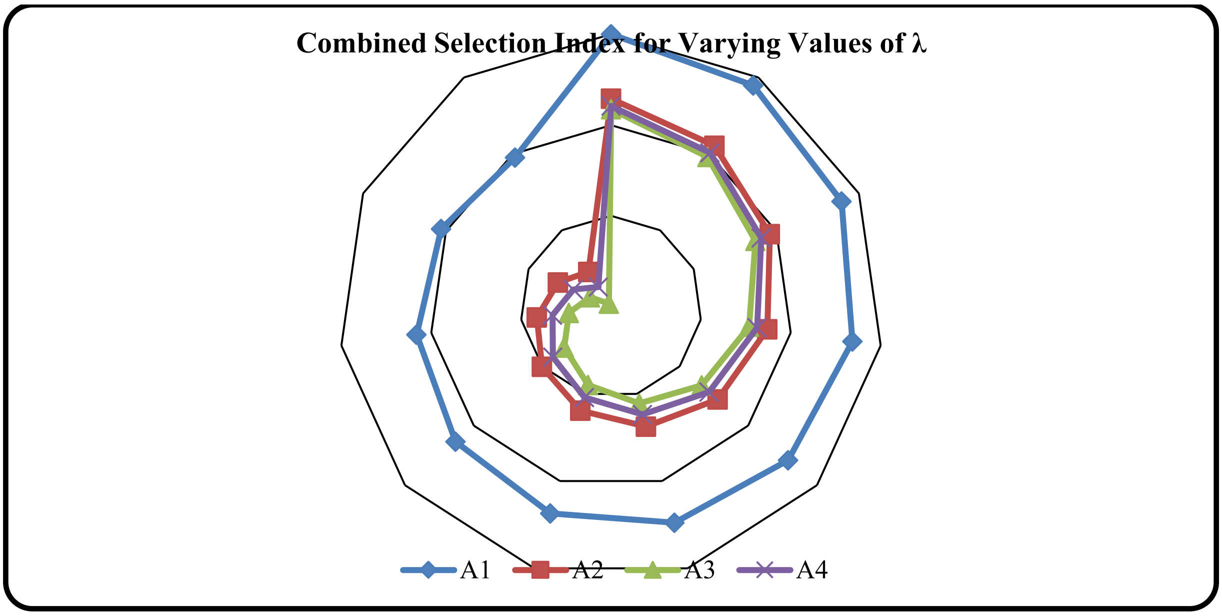

Fig.15. Combined selection indies for different values of λ.

Fig. 16 presents a radar diagram that depicts the ranking orders of four alternative warehouse locations (A_1, A_2, A_3, and A_4) as the parameter λ changes in discrete increments. The diagram shows how the ranking of these warehouse locations evolves with varying values of λ, ranging from 0 to 1 in steps of 0.1. The Table 22 lists the performance scores for each warehouse location at each value of λ. For instance, at λ = 0, A_1 has the highest score (1.0000), followed by A_2, A_3, and A_4. As λ increases, the performance scores of the warehouse locations decrease across all values of λ, with A_1 consistently maintaining the highest rank (rank 1), A_2 in second place (rank 2), A_4 in third (rank 3), and A_3 in the lowest position (rank 4). This trend persists throughout all the discretely changing values of λ. The radar diagram visually illustrates this relationship, with the values for each warehouse location plotted on the diagram's axes. The changing patterns of the values as λ increases provide insights into how sensitive the warehouse rankings are to fluctuations in λ, highlighting which warehouse locations are more resilient or sensitive to changes in operational parameters represented by λ. In summary, the radar diagram effectively communicates the shifting rankings of warehouse locations in response to changes in the parameter λ, providing a visual understanding of their comparative performance.

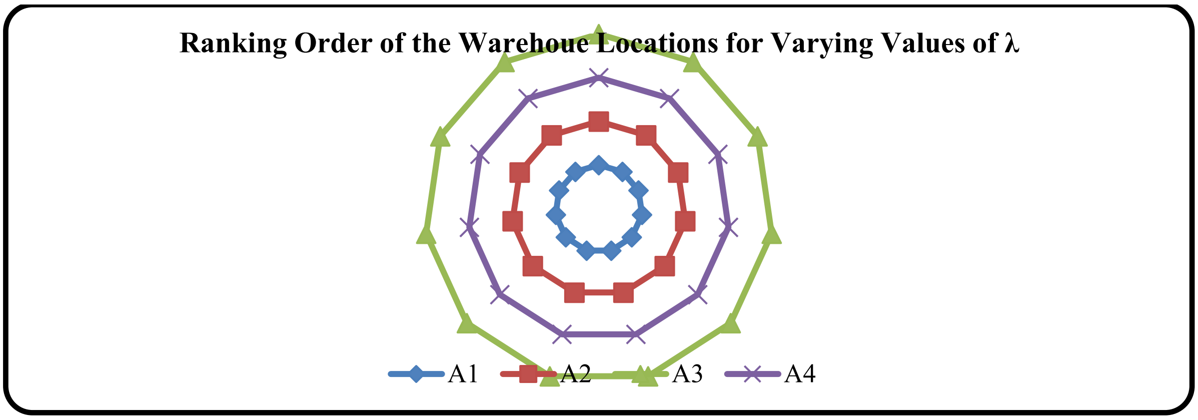

Fig.16. Ranking otders of the alternative warehouse locations for varying valus of λ.

6. Discussions

The proposed Fermatean fuzzy approach for enhanced decision-making in uncertain environments provides a structured framework for handling heterogeneous group dynamics. By leveraging linguistic assessments, Fermatean fuzzy sets, and aggregation operators, the methodology accommodates expert judgments, subjective preferences, and uncertainty inbuilt in the processes of decision-making.

The stepwise process outlined in the paper systematically guides decision-makers through problem definition, expert assessment, criteria weighting, alternative evaluation, and selection. By incorporating diverse perspectives from heterogeneous experts and employing Fermatean fuzzy techniques, the approach ensures robustness and adaptability across different decision contexts.

The utilization of linguistic variables, Fermatean fuzzy numbers, and aggregation techniques permits for demonstration of subjective judgments and uncertainty, enhancing the transparency and interpretability of decision outcomes. Moreover, the analysis of sensitivity conducted exhibits the robustness of the developed method to variations in decision-makers' preferences and model parameters, reaffirming its reliability and applicability in practical settings.

Overall, the proposed Fermatean fuzzy approach offers a valuable contribution to decision-making theory and practice, particularly in uncertain and heterogeneous environments. However, further research is needed to validate its effectiveness across a broader range of decision scenarios and to explore extensions and refinements to address specific challenges and limitations.

6.1 Comparison of the proposed and the existing methods

To compare the proposed Fermatean fuzzy based heterogeneous group TOPSIS algorithm with the existing method, let's create a tabular comparison based on various aspects of the methodologies, furnished in Table 23**.**

Table 23

Proposed Fermatean fuzzy based algorithm and existing methods

• **Aspect**	• Proposed Fermatean fuzzy based algorithm	• Existing Methods
• **Decision Criteria**	• Utilizes linguistic variables and Fermatean fuzzy sets for assessing criteria weights and decision matrix.	• May or may not incorporate linguistic variables and fuzzy sets in assessing criteria weights and decision matrix.
• **Expert Involvement**	• Forms a heterogeneous expert group and considers their assessments for criteria weights and decision making.	• May involve experts, but the extent of their involvement and the process may vary.
• **Weighting Method for Criteria**	• Uses Fermatean fuzzy entropy method and expert assessments for criteria weighting.	• May use various methods such as AHP, Entropy method, or direct expert judgments for criteria weighting.
• **Handling of Linguistic Variables**	• Incorporates linguistic variables for assessing performance ratings and judgment of criteria weights.	• May or may not explicitly incorporate linguistic variables for assessment.
• **Normalization of Decision Matrix**	• Normalizes the decision matrix using a specific equation.	• May or may not perform normalization of the decision matrix.
• **Aggregation Method for Decision Matrices**	• Aggregates Fermatean fuzzy decision matrices using FFWA operator.	• May use simple averaging or weighted averaging methods for aggregating decision matrices.
• **Ideal Solutions Calculation**	• Calculates FFPI and FFNI solutions based on specific equations.	• Ideal solutions calculation method may vary.
• **Alternative Ranking Method**	• Uses proximity coefficients for alternative ranking.	• May use closeness coefficients, relative closeness, or other ranking metrics.
• **Overall Decision Metric**	• Computes Combined Selection Index (CSI) for each alternative.	• May use a different overall decision metric for alternative evaluation.

• **Application Scope**	• Applied to a specific decision-making problem (e.g., warehouse location selection).	• Applicable to various decision-making problems.
• **Output Interpretation**	• Provides ranking order of alternatives, proper decision making, influential experts, and important criteria.	• Outputs may vary based on the specific method used.
• **Complexity and Computation Requirements**	• May have higher complexity due to the incorporation of linguistic variables, fuzzy sets, and multiple steps.	• Complexity may vary based on the method used, but typically simpler compared to methods involving fuzzy logic.

This comparison highlights the distinctive features and potential advantages of the proposed Fermatean fuzzy based algorithm over existing methods, particularly in terms of handling linguistic variables, incorporating expert assessments, and providing a comprehensive decision-making framework. However, the complexity and computation requirements of the proposed method may also be higher compared to existing methods.

6.2 Advantages of the Proposed Method

The Proposed Fermatean fuzzy based heterogeneous group TOPSIS algorithm presents several benefits over existing methods in the realm of Multiple Criteria Decision Making (MCDM). Here are some of the key advantages:

- Incorporation of Heterogeneous Expertise: This method allows for the formation of a heterogeneous expert group consisting of decision makers with diverse backgrounds and expertise. This ensures a comprehensive consideration of different perspectives and insights, leading to more robust decision-making outcomes compared to methods that rely on a homogeneous group of experts.
- *Linguistic Variables and Fermatean Fuzzy Sets:* By utilizing linguistic variables and Fermatean fuzzy sets, the proposed method offers a flexible framework for capturing subjective judgments and uncertainties in decision-making processes. This allows decision makers to express their preferences and perceptions in a more intuitive and nuanced manner, enhancing the accuracy and reliability of the decision outcomes.
- *Assessment of Expert Weight:* The method provides a systematic approach for assessing the weight of each expert based on meaningful factors or attributes. This ensures that the influence of each expert on the decision-making process is appropriately accounted for, leading to more balanced and informed decisions.
- *Integration of Expert Judgment on Criteria Weights:* Unlike some existing methods that rely solely on predefined weights for criteria, the proposed method incorporates expert judgment to determine the importance weight of each criterion. This allows decision makers to leverage their experience, analysis, and perception to assign weights that accurately reflect the relative significance of different criteria in the decision-making context.
- *Utilization of Fermatean Fuzzy Entropy Method:* The method employs Fermatean fuzzy entropy to further refine the determination of criteria weights, taking into account the degree of uncertainty and variability associated with each criterion. This enhances the robustness and reliability of the weight estimation process, leading to more accurate and meaningful decision outcomes.
- *Comprehensive Evaluation of Alternatives:* The method facilitates a comprehensive evaluation of alternatives by considering multiple criteria assessed by different experts using predefined linguistic variables. This ensures that the decision-making process takes into account a diverse range of factors and perspectives, leading to more holistic and informed decisions.
- *Aggregation of Decision Matrices*: Through the aggregation of Fermatean fuzzy decision matrices weighted by expert weights, the method integrates individual assessments into a single combined matrix. This allows decision makers to synthesize diverse opinions and preferences, leading to more consensus-driven and robust decision outcomes.
- *Flexible Selection Criteria:* The method offers flexibility in selecting the best alternative by providing multiple criteria for evaluation, such as proximity coefficient, closeness coefficient, and combined selection index. This allows decision makers to tailor the decision criteria according to the specific requirements and priorities of the decision-making context, leading to more customized and relevant decision outcomes.

Overall, the proposed Fermatean fuzzy based heterogeneous group TOPSIS algorithm offers a comprehensive and flexible approach to MCDM, integrating diverse expertise, subjective judgments, and uncertainties to facilitate more informed and robust decision-making processes.

7. Conclusions

This investigation introduces a novel approach, leveraging Fermatean fuzzy sets, to enhance decision-making within heterogeneous group dynamics, particularly focusing on the critical task of warehouse location selection in complex industrial scenarios. In the face of escalating uncertainties and evolving challenges characteristic of Industry 4.0, traditional decision-making methodologies often fall short, necessitating innovative solutions capable of navigating the intricacies of modern industrial landscapes.

Our research endeavors center on pioneering the development of a Fermatean fuzzy-based Multi-Criteria Group Decision-Making (FFMCGDM) technique, which extends beyond the constraints of conventional fuzzy sets. By integrating Fermatean principles with heterogeneous group dynamics, our approach offers a comprehensive framework tailored to address the multifaceted challenges inherent in decision-making processes within diverse group settings.

Through the fusion of individual preferences, expert opinions, and mathematical tools, our methodology facilitates the evaluation of various options while accounting for uncertainties and preferences within the group. Moreover, the introduction of a new measure, the proximity coefficient, addresses limitations of existing techniques, enhancing the robustness and effectiveness of our approach.

By demonstrating the application of our method in real-world scenarios and conducting sensitivity analyses under different conditions, we showcase its efficacy and adaptability in uncertain environments. We anticipate that the adoption of our approach will lead to improved decision quality, enhanced group collaboration, and better adaptation to changing industrial landscapes.

Moreover, we have addressed limitations observed in existing decision-making approaches, such as the incapacity of Hadi-Venchech and Mirjaabri's method to provide solutions in specific cases. Our corrective measures, including the introduction of the proximity coefficient, have enhanced the efficacy of our framework.

Furthermore, sensitivity analysis has validated the robustness of our proposed method across varying decision-makers' preferences, reaffirming its suitability for decision-making concerning warehouse location selection under broader uncertainty.

Looking ahead, future research efforts will focus on extending our analysis in several directions:

- Exploring alternative Multiple Criteria Decision Making (MCDM) tools, such as VIKOR, COPRAS, MULTIMOORA, and WASPAS, in conjunction with our methodology.
- Investigating the applicability of interval-valued and hesitant Fermatean fuzzy sets within our approach to further enhance its versatility.
- Addressing interdependent criteria and developing appropriate weight measuring techniques to refine the decision-making process.

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