Uncertain Supply Chain Management 1 (2013) 87–98

Contents lists available at GrowingScience

Uncertain Supply Chain Management

homepage: www.GrowingScience.com/uscm

Determining the periodicity and planned lead time in serial-production systems with dependent demand and uncertain lead time

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C H R O N I C L E

Article history:
Received January 12, 2013
Received in revised format
10 May 2013
Accepted June 19 2013
Available online
June 20 2013

Keywords:
Serial production systems
Planned lead time
Periodic order quantity
Uncertainty
Supply chain

A B S T R A C T

This paper considers a multi period serial production systems for one product and deals with the problem of planned lead-time calculation in a Material Requirement Planning (MRP) environment under probabilistic lead times. It is assumed that lead times for all stages have the same distribution with different parameters. A MRP approach with periodic order quantity (POQ) policy is used for the supply planning of components. The objective is to minimize the sum of fixed ordering, holding and backlogging costs. A mathematical model is suggested and then an optimal planning lead-time, ordering quantity and periodic time are determined.

1. Introduction

Material requirement planning (MRP) is a well-known approach inventory management of dependent demand items. We consider a multi stage, multi period serial-production system with constant demand and probabilistic lead-time. Examples of serial production systems include electronic, automotive assembly systems, and packing systems. Serial production systems are in a sense simpler form of multistage production systems. For the corresponding single level problem with random lead-time, Kaplan (1970) suggested a finite horizon dynamic programming model whose optimal inventory policy turned out to depend on whether ordering cost is fixed or not. A probabilistic model was developed for arrival of outstanding orders assuming that orders would not cross in time and that the arrival probabilities are independent of the number and size of outstanding orders. Based on those assumptions, it was demonstrated that the sequential multidimensional minimization problem normally associated with the random lead-time model could be reduced to a sequence of one-dimensional minimizations. These policies were shown to be quite similar to those obtained with

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doi: 10.5267/j.uscm.2013.06.002

deterministic lead times; some differences are noted in the behavior of the single-period critical numbers (when the setup cost is zero).

Yano (1987b) suggested an analytic approach and a single-period model to determine optimal planned lead times in serial production systems in which the actual procurement and processing times may be stochastic. The objective in this paper is to minimize the sum of inventory holding. A general solution is proposed for a two-stage serial system, which for most cost structures and lead time distributions, is a single-pass algorithm. The paper provides some insight into the characteristics of optimal safety time policies. In another paper, Yano (1987) investigated the problem of determining optimal planned lead times in serial production systems in which the actual procurement and processing times may be stochastic. The objective was to minimize the sum of inventory holding costs and job tardiness costs. Optimal lead times in serial production systems were also examined by various researchers (Gong et al., 1994; Mohan & Ritzman, 1998). Harrison and Lewis (1996) presented another related work on serial production systems, which includes lot sizing in serial systems. Conway et al. (1988) examine the role of work-in-process inventory in serial production systems. Ballou and Pazer (1982) and Rebello et al. (1995) examined inspection issues in serial production systems.

Tang (1990) presented a discrete time model of a multi-stage production system that faces two major types of uncertainties: the output rate at each production stage and the demand for the finished product. Tang proposed a scheme in which a complex production rule is approximated by a linear production rule. Wein (1992) studied on a make-to-order marketing environment where an order was met from a single production lot size. A Markov decision process model was developed and it was solved using dynamic programming techniques. The model assumes that demand is given, and material, processing and rework costs are linear in the production lot size. Modeling random yield at each stage of the production process is of key interest. The solution to the problem is characterized and the sensitivity of the solution for the parameters of the model is examined.

Karimi et al. (1992) analyzed scheduling of multistage serial production systems under constant demand and infinite horizon. An integer nonlinear programming formulation is presented for determining a stationary, cyclic schedule with no stock-outs in any inventory and minimum sum of setup and inventory costs. It allows a lot-sizing policy involving arbitrary, non-integer splitting/merging of lots. Three, almost optimal, heuristic algorithms and an exact branch and bound algorithm are developed using analytical results. Their evaluation using simulated problems shows the branch and bound algorithm to be the best, as it is fast even for systems with as many as 11 stages. Elhafsi (2002) considered a production system consisting of N processing stages which actual lead times at the stages are stochastic. The problem was formulated as a convex nonlinear programming problem. The latter was then solved using classical convex optimization algorithms. For the special case of exponentially distributed lead times, the objective function is derived in a closed form. The objective is to determine the planned lead times at each stage so as to minimize the expected total inventory costs, tardiness penalties, and a backlog penalty for not meeting demand due date at the last stage.

Hnaien et al. (2008) considered single period, multi stage problem of planned lead-time calculation in a Material requirement Planning (MRP) environment under stochastic lead times and used lot-for-lot policy for all levels. An optimization model was suggested for the serial production system. In the present paper, the criterion considered is the sum of backlogging and holding costs. Assume that the actual level lead times are independent discrete random variables. The distribution probability for the different levels cannot be identical. We use POQ policy, which is a lot size technique that orders to cover requirements for a variable number of periods based on order and holding costs, as opposed to a fixed period quantity that uses standard number of periods. We assume that a FIFO policy is used to fill the backorders once the production run starts.

2. Problem formulation

In this section, we consider a serial production system with one type of product. This production system consisting of m processing stages which actual lead times at the stages are probabilistic and the POQ policy is used for this multi stage multi period system. We suppose that the demand for finishing goods is constant for each period, but actual lead-time is probabilistic for all stages. If at a certain stage, a job (or a batch) is completed before its planned lead-time, it is held at that stage until its planned release time, including inventory-holding cost. Similarly, if at a certain stage, a job is not received as planned, a penalty cost is incurred at that stage. We assume that stage m receives its job as planned, therefore, incurs no tardiness penalty. In addition, Stage 1 will incur a backlog penalty if it completes processing a job later than the planned time. This penalty represents a customer goods will lose and cost of delayed delivery to the customer and if Stage 1 completed before the due date, the final product held until delivering it to the customer. The policy of ordering is periodic order quantity (POQ). Final products are ordered at every P period. We suppose that we have a demand D for finished products at each period with a fixed due date. To satisfy this, we need to launch the production processes composed of m serial levels for the lot of D items.

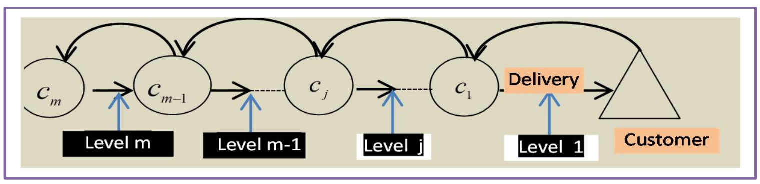

Fig. 1. An m-level linear supply chain

The raw materials are released at level *m*, the semi-finished products are processed at levels $m_1, m_2, \ldots$ and finally, the finished product is produced at level 1. We assume that lead-time at each level is probabilistic. The objective is to find the component release dates at each level and time period (P) for minimizing the sum of the holding costs for the components of each level and backordering cost, lost sale cost, holding cost for the finished product.

t : Index of period's $t=1, 2, 3$,
A : ordering cost per order,
D : Demand (known) for finished product for at the period t
p : Time periods for each ordering
h : Per unit holding cost per time unit
$\hat{\pi}$: Per unit backorder cost per time unit
l_i : Random lead time for level i
$l = \sum_{i=1}^{N} l_i$: Total lead-time of the system
x_i : Planned lead time for level i
$x = \sum_{i=1}^{N} x_i$: Total planned lead-time of the system (planned lead-time for finished product)
$f(l_i)$: The continue distribution of lead-time in level i
$f(l)$: The convolution of lead-time with continue distribution
$P(l_i = j)$: The discrete distribution of lead time distribution in level i
$P(L = j)$: The convolution of lead-time with discrete distribution

Variables

P: periodicity
X: planned lead time for finished product

2.1 convolution

The probability distribution of the sum of two or more independent random variables is the convolution of their individual distributions. The term is motivated by the fact that the probability mass function or probability density function of a sum of random variables is the convolution of their corresponding probability mass functions or probability density functions respectively.
Many well-known distributions have simple convolutions.

Convolution defecation:

Let $l_1, l_2, \ldots, l_N$ be independent and identically distributed random variables with the common distribution function F and probability density function f. then the distribution function of the sum L_n is the n-fold convolution of itself F such as

$$F^{n*}(x) = F^{(n-1)*} * F(x) \qquad (n \geq 2) \tag{1}$$

where $F^{1*}(x) = F(x)$ and its probability density function is

$$f^{n*}(x) = f^{(n-1)*} * f(x) \qquad (n \geq 2) \tag{2}$$

where

$$f^{1*}(x) = f(x) \ . \tag{3}$$

Convolution of Uniform Distributions

Theorem 1: Let l_i be independent random variable with PDF $f_{l_i} = \frac{1}{b_i - a_i}, i = 1,2,\ldots,N \geq 2$ then PDF $l = \sum_{i=1}^{N} l_i$ is given as follow:

$$f_l(l) = \begin{cases} f_l^n(l) & \text{if} \quad A \leq l \leq B \\ 0 & \text{Otherwise} \end{cases} \tag{4}$$

where $A = \sum_{i=1}^{N} a_i \quad B = \sum_{i=1}^{N} b_i$, $A_n = \prod_{i=1}^{n} (b_i - a_i)$, and

$$f_n(l) = \frac{1}{(n-1)!A_n}(-1)^R \sum_{k=1}^{n} \sum_{m=k}^{n} \left(\left(\left(x - \sum_{i=k}^{m} a_i - \sum_{j=1 \neq i}^{n} b_j \right)^+ \right)^{n-1} \right) + \frac{1}{(n-1)!A_n}(-1)^R \left(\left(x - \sum_{j=1}^{n} b_j \right)^+ \right)^{n-1} . \tag{5}$$

R is the total number of b_j , j=1, 2…N

The proof of Eq. (5) is shown in the Appendix A. List of convolutions of some of probability distributions are shown in Appendix B

3. Model development

The lead-time is assumed probabilistic. The planned lead-time is x_i for level i. The orders for products are made at the beginning of the periods 1, p+1, 2p+1… and there is no order made in other periods. Order quantities are constant and equal to PD (P is a decision variable). (See Fig. 2)

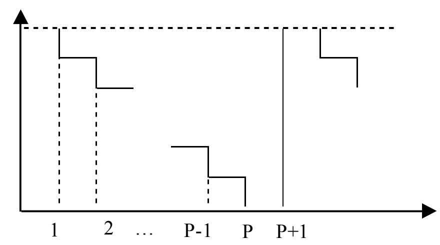

Fig.2. An illustration of the planning problem when planned lead time equal to actual lead time

Because of probabilistic lead-time, there are three states in action:

The planned lead time equal to actual lead time (see Fig. 1). This state has not backorder and model cost is equal to:

$$C_1(x,p) = \left[A + (p-1)hD + (p-2)hD + \ldots + 2hD + hD\right] \times f(l = x) = \left[A + \frac{p(p-1)}{2}hD\right] \times f(l = x) \tag{6}$$

where $f_t(l)$ is the convolution of lead-time. The planned lead-time is smaller than to actual lead-time (see Fig. 2). If the finished product is assembled after the due date, there exists backlog.
In this state, the cost is equal to:

$$C_2(x,p) = \left[A + \frac{\left(p-(l-x)\right)\left(p-1-(l-x)\right)}{2}hD\right] \times P(l > x) + \left[bD\frac{(l-x)(l-x+1)}{2}\right] \times P(l > x) \tag{7}$$

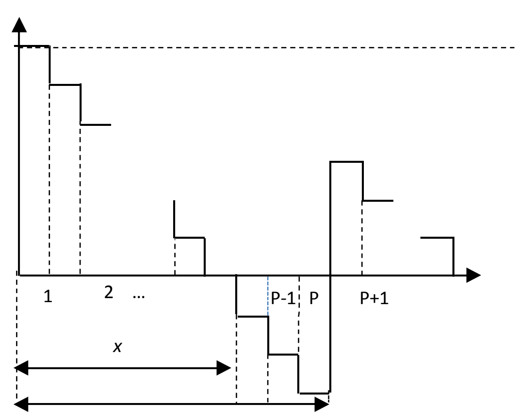

Fig .3: An illustration of the planning problem when planned lead-time is smaller than to actual lead-time

The planned lead time is bigger than the actual lead time (see Fig. 3). In this state, the cost is equal to:

$$C_3(x,p) = \left[A + \frac{p(p-1)}{2}hD + hpD(x-l)\right] \times P(l < x) \tag{8}$$

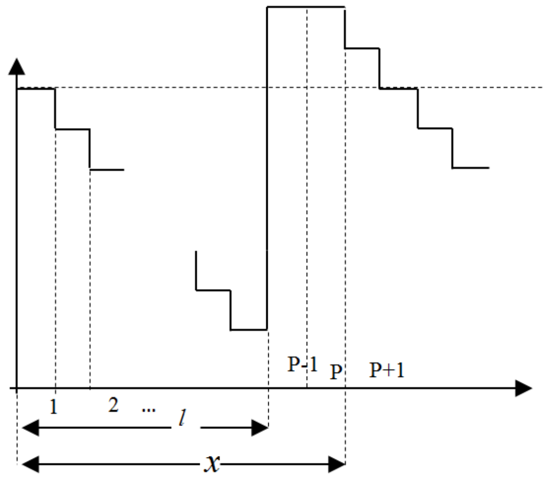

Fig. 4. An illustration of the planning problem when planned lead-time is bigger than the actual lead-time

Objective function for continue distribution

By using Eqs. (6-8), total cost can be expressed as follows:

$$C(x,p)=C_1(x,p)+C_2(x,p)+C_3(x,p)=A+\frac{p(p-1)}{2}hD+\big(hpD(x-l)\big)\times P(l<x)+\left(bD\frac{(l-x)(l-x+1)}{2}\right)\times P(l>x)-\frac{hD(l-x)(2p-1)}{2}\times P(l>x) \tag{9}$$

Then by using Eq. (9), the expressed unit cost will be as follows:

$$\hat{C}(x,p)=\frac{C(x,p)}{p}$$

$$\hat{C}(x,p)=\frac{A}{p}+\frac{(p-1)}{2}\times h\times D+h\times D\times E\big[(x-l)\big]+\frac{D}{2p}(h+b)\times\int_{l\geq x}(l-x)^2 f(l)d_l+\frac{D}{2p}(h+b)\times\int_{l\geq x}(l-x)f(l)d_l \tag{10}$$

Objective function for discrete distribution

Then by using Eq. (9), unit cost for discrete distribution is expressed as follows:

$$\hat{C}(x,p)=\frac{A}{p}+\frac{(p-1)}{2}\times h\times D+h\times D\times E\big[(x-l)\big]+\frac{D}{2p}(h+b)\times\sum_{l>x}(l-x)^2 p(L=l)+\frac{D}{2p}(h+b)\times\sum_{l>x}(l-x)p(L=l) \tag{11}$$

4. Solution procedure

Theorem 2: The objective function $\hat{C}(x,p)$ in Eq. (10) is convex.

Proof:

The function $\hat{C}(x,p)$ is convex if and only if twice-differentiable function is non-negative.

$$\frac{\partial^2\hat{C}(x,p)}{\partial x^2}\geq 0 \quad \text{for all } x\in dom\,\hat{C}(x,p) \tag{12}$$

In addition, the function $\hat{C}(x,p)$ is strictly convex if twice-differentiable function is positive

$$\frac{\partial^2\hat{C}(x,p)}{\partial x^2}>0 \quad \text{for all } x\in dom\,\hat{C}(x,p) \tag{14}$$

$$\frac{\partial\hat{C}(x,p)}{\partial x}=-\frac{D}{p}(h+b)\times\int_{l\geq x}(l-x)f(l)d_l-\frac{D}{2p}(h+b)\times\int_{l\geq x}f(l)d_l \tag{15}$$

$$\frac{\partial^2 \hat{C}(x,p)}{\partial x^2} = \frac{D}{p}(h+b) \times \int_{l \geq x} f(l) d_l \tag{16}$$

The Eq (10) is positive then, according to Eq (13) the objective function is strictly convex.

Optimal solution for continues distribution

To find the optimal planned lead-time (x) and the optimal periodicity (P), we use differentiate $\hat{C}(x,p)$ with respect to x and solve the resulted system of equation obtained by equating the derivative to zero (see Eq. (13)).

$$\frac{\partial \hat{C}(x,p)}{\partial x} = -\frac{D}{p}(h+b) \times \int_{l \geq x} (l-x) f(l) d_l - \frac{D}{2p}(h+b) \times \int_{l \geq x} f(l) d_l \quad \frac{\partial \hat{C}(x,p)}{\partial x} = 0 \Rightarrow \int_{l \geq x} (l-x+.5) f(l) d_l = \frac{ph}{h+b} \tag{16}$$

To find p, we let it to have respectively the values 1, 2, and obtain by Eq. (16) the optimal x for each value p and x that cause the lowest cost, will be the optimum solution. In the model considered, the demand D is constant and the quantities ordered are the same and equal to D_p; so the optimal planned lead time x is also the for all orders. Noted that x is planned lead time for pD items and l is actual lead time for D items then in the all equation x equal to x_p/p which is planned lead time p periods.

Optimal solution for discrete distribution

To find the optimal planned lead-time (x) and the optimal, we use $\Delta\hat{C}(x,p) = \hat{C}(x,p) - \hat{C}(x-1,p) \geq 0$ or $\Delta\hat{C}(x,p) = \hat{C}(x+1,p) - \hat{C}(x,p) \leq 0$.

Theorem 3: the optimal solution for discrete distribution is obtained by:

$$\sum_{l>x} (l-x-1) p(L=l) \leq \frac{h \times p}{h+b} \tag{17}$$

The prove of this Theorem is expressed in Appendix C

5. Numerical example

Consider the following data: N= 3, h = 10, b = 100, A=100, D = 10. The probability distribution for all stages is uniform and as follow:

Table 1
The example input data

i		Distribution	a_i	b_i
1	$g(l_1)$	uniform	4	6
2	$g(l_2)$	uniform	2	5
3	$g(l_3)$	uniform	5	10

By of Eq (5) $f_3(l)$ equal to:

$$f_3(x) = \begin{cases} \frac{1}{60} \times (x^2 - 22x + 121) & 11 \leq x < 13 \\ \frac{1}{60} \times (4x - 48) & 13 \leq x < 14 \\ \frac{1}{60} \times (-x^2 + 32x - 244) & 14 \leq x < 18 \\ \frac{1}{60} \times (-4x + 80) & 18 \leq x < 19 \\ \frac{1}{60} \times (x^2 - 42x + 441) & 19 \leq x < 21 \end{cases}$$

The probability distributions of the lead times are reported in Table 1.
According to Eq (16) have:

$$\int_{l\geq x} (l-x+.5)f(l)d_l = \frac{ph}{h+b} = \int_{l\geq x} (l-x+0.5)\frac{1}{60}\left(\frac{1}{60}\left[\left((l-11)^+\right)^2 - \left((l-13)^+\right)^2 - \left((l-14)^+\right)^2\right.\right.$$

$$\left.\left. - \left((l-16)^+\right)^2 + \left((l-16)^+\right)^2 + \left((l-18)^+\right)^2 + \left((l-19)^+\right)^2 - \left((l-21)^+\right)^2\right]\right)d_l = \frac{p\times 10}{10+b}$$

After that, the optimal lead-times x are obtained by using (16), for each periodicity p. The results are reported in Table 2.

Table 2
The example output data

P	1	2	3	4	5	6
X	18.547	**18.007**	17.6320	17.332	17.0770	16.853
x_p	18.547	**36.014**	52.896	69.328	85.385	101.118
$\hat{C}(x,p)$	409.458	**367.5248**	372.052	390.976	416.8195	446.7321

The global optimal solution is obtained when p=2 and x=18.007. The minimum cost is 367.5248. Note that for Lot for Lot policy (p=1) the cost is 409.458. The optimal solution is p=2, $x^*=18.007$ and $C(c, p)^* = 367.5248$. The order quantity is as follows,

$$Q^* = p\times D = 2\times 10 = 20$$

The function cost is shown in Fig 5.

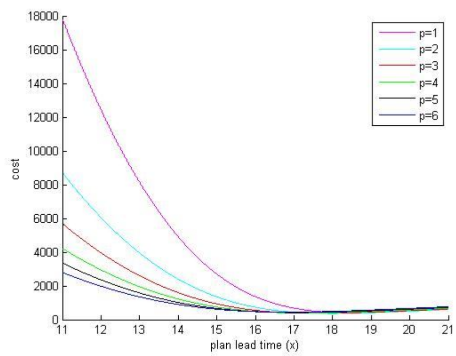

Fig. 5. The cost functions for different values of p

The answer of this system is dependent to the cost parameters. For example if setup cost was very small rather than holding cost therefore the lot for lot ordering system is better. Table 3 show the optimal solution for varies parameter's cost.

Table 3
The optimal solution for varies parameter's cost

P. No	A	h	b	p^*	x^*	$x_p^* = p\times x^*$	$C(x^*, p^*)$
1	100	10	100	2	18.007	36.014	367.5248
2	**500**	10	100	5	17.077	85.385	496.8195
3	**1000**	10	100	6	16.9	101.4	596.7
4	**10000**	10	100	18	1.5	27	1457
5	100	**20**	100	2	17.4260	34.852	596.1508
6	100	**40**	100	2	16.8	33.6	952.5
7	100	**100**	100	3	15.2	45.6	1668.8
8	100	**1000**	100	6	11	66	5418
9	100	10	**200**	2	18.515	37.03	406.8962
10	100	10	**400**	2	18.946	37.892	440.7579
11	100	10	**1000**	2	19.416	38.832	477.6658
12	100	10	**10000**	2	20.199	40.398	538.792

According to the results of Table 3, with increase in the setup cost, the periodic time is increased, but the planned lead time is reduced. With increase in holding cost, the periodic time is increased, but the planned lead time is reduced. And with increase in back order cost, the periodic time is fixed, but the planned lead time is increased.

6. Conclusion

The goal of this paper was to present a model for optimizing the planned lead-time and order periodicity for production and assembly systems with random stage procurement times. The proposed model and algorithms minimize the sum of the average holding cost, backlogging product and setup cost. We have assumed that the distributions of lead times in each stage are the same but these parameters are different. By using of convolution concept, the distribution lead-time was determined. Lead time distribution can be having continue or discrete distribution. Therefore we find a close form equation for continue and discrete distribution and by using this equation optimal planned lead-time. This method, also can calculate the cost of the Lot for Lot policy. The cost of Lot for Lot order policy is when P equal to one. In this paper, various problems have been solved to show the efficacy of cost parameter's on optimal planned lead-time and periodicity time.

References

Ballou, D.P., & Pazer, H.L. (1982).The impact of inspector, fallibility on the inspection policy in serial production systems. *Management Science*, 28, 387-399.

Conway, R., Maxwell, W., McClain, J.O., & Thomas, L.J. (1988).The role of work-in-process inventory in serial production lines. *Operations Research*, 36, 229-241.

Elhafsi, M. (2002). Optimal lead times planning in serial production systems with earliness and tardiness costs. *IIE Transactions*, 34, 233-243.

Gong, L., de Kok, T., & Ding, J. (1994). Optimal lead times planning in a serial production system. *Management Science*, 40, 629-632.

Harrison, T.P., & Lewis, H.S. (1996). Lot sizing in serial assembly systems with multiple constrained resources. *Management Science*, 42, 19-36.

Hogg, R. V., McKean, J. W., & Craig, A. T. (2013). *Introduction to mathematical statistics*. Pearson.

Hnaien, F., Dolgui, A., & Louly, M. A. O. (2008). Planned lead time optimization in material requirement planning environment for multilevel production systems. *Journal of Systems Science and Systems Engineering*,*17*(2), 132-155.

Kaplan, R. S. (1970). A dynamic inventory model with stochastic lead times.*Management Science*, *16*(7), 491-507.

Karimi, I.A. (1992). Optimal cycle times in multistage serial systems with set-up and inventory costs. *Management Science*, 38, 1467-1481.

Mohan, R.P., & Ritzman, L.P. (1998). Planned lead times in multistage systems. *Decision Sciences*, 29, 163-191.

Rebello, R., Agnetis, A., & Mirchandani, P.B. (1995). Specialized Inspection Problems in Serial Production Systems. *European Journal of Operational Research*, 80, 277-296

Sheldon, R. (2002). *A First Course In Probability, 6/E*. Pearson Education India.

Yano, C. A. (1987b). Planned leadtimes for serial production systems. *IIE transactions*, *19*(3), 300-307.

Tang, C. S. (1990). The impact of uncertainty on a production line. *Management Science*, *36*(12), 1518-1531.

Wein, A. S. (1992). Random yield, rework and scrap in a multistage batch manufacturing environment. *Operations Research*, *40*(3), 551-563.

Yano, C.A. (1987). Setting planned lead-times in serial production systems with tardiness costs. *Management Science*, 33, 95-106.

Appendix A. proof Eq. (5)

Proof:

For prove this theorem, using of Mathematical induction.

Without loss of generality, we assume that $l_i \sim U(a_i, b_i)$ then $F(l_i)$ is:

$$F_{l_i}(x) = \frac{(x-a_i)^+ - (x-b_i)^+}{b_i - a_i}, \tag{18}$$

where $(x-b_i)^+ = Max\left((x-b_i)^+, 0\right)$. In addition, we note that, by convolution, probability density functions and distribution functions are related as follows:

$$f_{n+1}(x) = \int_{a_{n+1}}^{b_{n+1}} f_n(x-y) f_{x_{n+1}}(y) d_y = \frac{F_n(x-a_{n+1}) - F_n(x-b_{n+1})}{b_{n+1} - a_{n+1}}. \tag{19}$$

Claim 1.2 One has

$$F_1(x) = \frac{(x-a_1)^+ - (x-b_1)^+}{b_1 - a_1}, \tag{20}$$

and

$$f_2(x) = \frac{(x-a_1-a_2)^+ - (x-a_2-b_1)^+ - (x-a_1-b_2)^+ + \left(x-(b_1+b_2)\right)^+}{(b_1-a_1)(b_2-a_2)}. \tag{21}$$

Claim 1.3 One has

$$F_2(x) = \frac{\left((x-a_2-a_1)^+\right)^2 - \left((x-a_2-b_1)^+\right)^2 - \left((x-b_2-a_1)^+\right)^2 + \left((x-(b_1+b_2))^+\right)^2}{2A_2}. \tag{22}$$

and

$$f_3(x) = \Big(\left((x-a_1-a_2-a_3)^+\right)^2 - \left((x-b_1-a_2-a_3)^+\right)^2 - \left((x-b_2-a_1-a_3)^+\right)^2 + \left((x-(b_1+b_2+a_3))^+\right)^2$$
$$- \left((x-a_2-a_1-b_3)^+\right)^2 + \left((x-a_2-b_1-b_3)^+\right)^2 + \left((x-b_2-a_1-b_3)^+\right)^2 - \left((x-(b_1+b_2+b_3))^+\right)^2 \Big/ 2\times A_3 \tag{23}$$

Then according to Mathematical induction theory the Eq. (25) holds for n=k

$$f_k(l) = \frac{1}{(k-1)!A_k}(-1)^R \sum_{j=1}^{k} \sum_{m=j}^{k} \left(\left(\left(x - \sum_{i=j}^{m} a_i - \sum_{f=1\neq i}^{k} b_f\right)^+\right)^{k-1}\right) + \frac{1}{(k-1)!A_k}(-1)^R \left(\left(x - \sum_{j=1}^{k} b_j\right)^+\right)^{k-1} \tag{24}$$

$$F_{k-1}(l) = \frac{1}{(k-1)!A_{k-1}}(-1)^R \sum_{j=1}^{k-1} \sum_{m=j}^{k-1} \left(\left(\left(x - \sum_{i=j}^{m} a_i - \sum_{f=1\neq i}^{k-1} b_f\right)^+\right)^{k-1}\right) + \frac{1}{(k-1)!A_{k-1}}(-1)^R \left(\left(x - \sum_{j=1}^{k-1} b_j\right)^+\right)^{k-1}. \tag{25}$$

Then we should prove the equation is true for n=k+1.

We proceed by induction. Claims 1.2 and 1.3 show that Eqs. (24) and (25) hold for n = 1 and n = 2. Let us now assume that they hold for n = k and prove that they also hold for n = k+1. To this purpose Eq. (26) with n = k+1 then follows. To obtain Eq. (27) for n = k we use of Eq(24) and of the just obtained expression of $F_k(l)$. Hence,

$$f_{k+1}(l)=\frac{1}{(k)!A_{k+1}}(-1)^R\sum_{j=1}^{k+1}\sum_{m=j}^{k+1}\left(\left(\left(x-\sum_{i=j}^{m}a_i-\sum_{f=1\neq i}^{k+1}b_f\right)^+\right)^k\right)+\frac{1}{(k)!A_k}(-1)^R\left(\left(x-\sum_{j=1}^{k+1}b_j\right)^+\right)^k \tag{26}$$

$$F_k(l)=\frac{1}{(k)!A_k}(-1)^R\sum_{j=1}^{k}\sum_{m=j}^{k}\left(\left(\left(x-\sum_{i=j}^{m}a_i-\sum_{f=1\neq i}^{k}b_f\right)^+\right)^k\right)+\frac{1}{(k)!A_k}(-1)^R\left(\left(x-\sum_{j=1}^{k}b_j\right)^+\right)^k . \tag{27}$$

Appendix B. List of convolutions some of probability distributions (Sheldon, 2002; Hogg et al., 2013).

In probability theory, the probability distribution of the sum of two or more independent random variables is the convolution of their individual distributions. The term is motivated by the fact that the probability mass function or probability density function of a sum of random variables is the convolution of their corresponding probability mass functions or probability density functions respectively. Many well-known distributions have simple convolutions. The following is a list of these convolutions. Each statement is of the form $\sum_{i=1}^{n} x_i = y$ where $x_1, x_2, \ldots, x_n$ are independent and identically distributed random variables. In place of x_i and y the names of the corresponding distributions and their parameters have been indicated.

Discrete distributions

$$\sum_{i=1}^{n} Bernoulli\ (p) \sim Binomial\ (n,p) \quad 0<p<1 \quad n=1,2,\ldots \tag{28}$$

$$\sum_{i=1}^{n} Binomial\ (n_i,p) \sim Binomial\ (\sum_{i=1}^{n} n_i,p), \quad 0<p<1 \quad n=1,2,\ldots \tag{29}$$

$$\sum_{i=1}^{n} NegativeBinomial\ (n_i,p) \sim NegativeBinomial\ (\sum_{i=1}^{n} n_i,p) \quad 0<p<1 \quad n=1,2,\ldots \tag{30}$$

$$\sum_{i=1}^{n} Geometric(p) \sim NegativeBinomial\ (n,p) \quad 0<p<1 \quad n=1,2,\ldots \tag{31}$$

$$\sum_{i=1}^{n} Poisson(\lambda_i) \sim Poisson(\sum_{i=1}^{n}\lambda_i) \quad \lambda_i>0 \tag{32}$$

Continuous distributions

$$\sum_{i=1}^{n} Normal(\mu_i,\sigma_i^2) \sim Normal(\sum_{i=1}^{n}\mu_i,\sum_{i=1}^{n}\sigma_i^2) \qquad -\infty<\mu_i<\infty \tag{33}$$

$$\sum_{i=1}^{n} Gamma(\alpha_i,\beta) \sim Gamma(\sum_{i=1}^{n}\alpha_i,\beta) \qquad \alpha_i>0, \quad \beta>0 \tag{34}$$

$$\sum_{i=1}^{n} Exponential(\theta) \sim Gamma(n,\theta) \qquad \theta>0, n=1,2,\ldots. \tag{35}$$

$$\sum_{i=1}^{n}\chi^2(\lambda_i)\sim\chi^2(\sum_{i=1}^{n}\lambda_i)\quad \lambda_i=1,2,... \tag{36}$$

$$\sum_{i=1}^{n}cauchy(\alpha_i,\gamma_i)\sim cauchy(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\gamma_i)\qquad -\infty<\alpha_i<\infty\quad,\qquad \gamma_i>1 \tag{37}$$

Appendix C: The prove of Eq.(17)

By using Eq. (11), $\Delta\hat{C}(x,p)$ is expressed as follow:

$$\Delta\hat{C}(x,p)=\hat{C}(x+1,p)-\hat{C}(x,p)=h\times D+\frac{D}{2p}(h+b)\times\sum_{l>x}\left((l-x-1)^2-(l-x)^2\right)p(L=l)$$
$$+\frac{D}{2p}(h+b)\times\sum_{l>x}\left((l-x-1)-(l-x)\right)p(L=l)$$

$$\Rightarrow\Delta\hat{C}(x,p)=h\times D-\frac{D}{2p}(h+b)\times\sum_{l>x}\left((2\times(l-x)-1)\right)p(L=l)-\frac{D}{2p}(h+b)\times\sum_{l>x}p(L=l)$$
$$=h\times D-\frac{D}{p}(h+b)\times\sum_{l>x}\left((l-x-1)\right)p(L=l)$$

$$\Delta\hat{C}(x,p)\ge 0\Rightarrow h\times D-\frac{D}{p}(h+b)\times\sum_{l>x}\left((l-x-1)\right)p(L=l)\ge 0\Rightarrow\sum_{l>x}\left((l-x-1)\right)p(L=l)\le\frac{h\times p}{h+b}$$