When operating in highly competitive business environments, contemporary manufacturing firms must persistently find ways to fulfill timely orders with quality ensured merchandise, manage the unanticipated fabrication disruptions, and minimize total operating expenses. To address the aforementioned concerns, this study explores the optimal runtime decision for a manufacturing system featuring an expedited fabrication rate, random equipment failures, and scrap. Specifically, the proposed study considers an expedited rate that is linked to higher setup and unit costs. The fabrication process is subject to random failure and scrap rates. The failure instance follows a Poisson distribution, is repaired right away, and the fabrication of interrupted batch resumes when the equipment is restored. The defective goods are identified and scrapped. Mathematical modeling and optimization method are used to find the total system cost and the optimal runtime of the problem. The applicability and sensitivity analyses of research outcome are illustrated through a numerical example. Diverse critical information regarding the individual/joint impacts of variations in stochastic time-to-failure, expedited rate, and random scrap on the optimal runtime decision, total system expenses, different cost components, and machine utilization, can now be revealed to assist in in-depth problem analyses and decision makings.

1. Introduction

A manufacturing runtime problem with an expedited fabrication rate, random failures, and scrap is investigated in this study. Taft (1918) is believed to be the first who proposed a mathematical approach to calculate the most economical production lot (or called the economic production quantity (EPQ) model), by balancing costs of setup and holding to decide the lot size per cycle which minimizes total relevant expenses. A perfect manufacturing process with finite fabrication rate is assumed in the original EPQ model. But, in real production environments, due to varied unexpected situations, fabrication of defective/scrap goods and machine failure instances are inevitable. Literatures on production inventory systems with defective/scrap items are surveyed as follows: Schwaller (1988) examined three separate inventory models; namely, the basic economic order quantity (EOQ) model, EOQ with non-instantaneous replenishment, and EOQ with backlogging, by incorporating the product screening cost into total inventory relevant costs. Effects of defective proportion and inspection cost on these models were
investigated to obtain in-depth information on inventory policy making. Richter (1996) considered an EOQ model with the repairable and scrapped items using two preset variables $n$ and $m$ to control production and repair mode within a finite interval of time. The author started with the assumption of a constant scrap rate and cost minimum setup numbers to derive the minimum cost. Then, the author treated the optimal/minimum cost as a function of scrap rate to explore the convex/concave relationships between the cost and the scrap rate, and further discussed the optimality on cost. Konstantaras et al. (2007) studied the imperfect EOQ/EPQ system with an in-house 100% product screening process. In the end of inspection, two options were proposed to handle the defective goods: (i) to sell them at a discount price to a secondary marketplace; and (ii) to repair them completely with an extra rework cost per item. For both cases, the perfect goods are transported to warehouse in equal-size shipments. Their objectives were to not only retain the product quality, but also to decide the optimal lot-size and frequency of shipments per order that kept the system cost at minimal. A numerical illustration with sensitivity analysis showed how their system works. Extra works (Porteus, 1986; So & Tang, 1995; Grosfeld-Nir & Gerchak, 2002; Giri & Chakraborty, 11; Abubakar et al., 2017; Chiu et al., 2017; Shakoor et al., 2017; Sher et al., 2017; Chiu et al., 2018a,b; Gan et al., 2018; Imbachi et al., 2018; Pearce et al., 2018) focused on fabrication/inventory systems with different features of nonconforming goods and their consequent treatments.

Due to different unpredicted reasons in real manufacturing process, random machine failure is inevitable. Groenevelt et al. (1992) explored the optimal fabrication lot-size problem considering equipment failures along with two distinct controlling disciplines on breakdown corrections; namely, the no-resumption (NR) and the abort-resume (AR) disciplines. According to the NR-discipline, the fabrication of interrupted lot is not resumed after a failure occurs. While under the AR-discipline, if the current stock level falls below a preset threshold point, fabrication of the interrupted lot is resumed right away, when the equipment is fixed. Analyses, solution procedures, outcomes, and impacts of these disciplines on the optimal lot-size were separately carried out, illustrated, and discussed. Kuhn (1997) examined a dynamic batch size problem featuring exponential breakdowns. Two distinct scenarios were investigated. The first scenario assumed the setup is completely lost after a machine failed, and the second scenario assumed that the resuming setup cost is significantly lower than the standard setup expense, when the fabrication of interrupted batch is resumed. The author showed that under scenario one, if the production planner ignores the factor of machine failures the cost penalty will be remarkably higher than that of the classic EPQ model. Besides, the author recommended a conditional resumption idea for scenario two and suggested an approach by using dynamic programming for finding optimal lot-size solutions for both decisions for both scenarios. Dehayem Nodem et al. (2009) considered the production rate and repair/replacement decision makings for a fabrication system featuring random breakdowns. Consequent actions/policies immediately after a breakdown instance include (i) machine is under repair, or (ii) an identical spare machine is used. Their objective was to choose the optimal manufacturing rate and the repair/replacement policy that keep the long-run system expenses at minimum. A semi-Markov decision model (SMDM) was employed to help first decide the best repair/replacement policy, then, based on this policy the production rate was determined. By the use of numerical methods, the authors showed the optimality conditions, and revealed that as the number of machine failures rises, the on-hand stock level must be adjusted higher to avoid shortage. Chakraborty et al. (2013) studied an EMQ system which is subject to stochastic failure, repair, and stock threshold level (STL). The fabrication rate was considered as a decision variable and it links to the machine failure rate. The authors allocate additional capacity to hedge against random breakdowns. According to general distributions of breakdown and repair time, the basic model was built and two computational algorithms were proposed to help solve the optimal fabrication and STL that keep the long-run system expenses at minimalum. Extra works (Berg et al., 1994; Giri & Dohi, 2005a; Dahane et al., 2012; Liu et al., 2017a; Muzamil et al., 2017; Vujosevic et al.,
To shorten manufacturing completion time to fulfill buyer’s timely orders, production managers often expedite the fabrication rate. Arcelus and Srinivasan (1987) studied an EOQ-based system with several optimizing measures and different demands and prices. With the aim of making profit, the authors established decision rules to manage inventories of finished stocks, defined price using a markup rate of cost, and assumed demand as price-dependent function. Both order quantity and markup rate are decision variables, and optimization of their system was determined by three broadly used performance evaluators, which include profits, return on investment, and residual income. Balkhi and Benkherouf (1996) examined an inventory system featuring deteriorating products, time-varying demand and fabrication rates. A precise method was presented to derive the optimal stock refilling schedule for the proposed inventory system and the method was illustrated via a numerical example. Gharbi et al. (2006) investigated an unreliable multiproduct multi-machine fabrication system with adjustable fabrication rates and setup actions. Different setup times and costs are linked to each switching process, no matter it is a product- or machine-type of switch. Their objective was to minimize the overall operating costs by deciding the best fabrication rates and the optimal sequence of setups. The authors used the following approaches to solve this complex problem: (i) the stochastic optimal control theory, (ii) experimental design, (iii) discrete event simulation, and (iv) response surface method. Two different cases were studied: case one considered single machine unreliable system featuring exponential breakdown and repair time distribution, and case two considered five machines system featuring non-exponential breakdown and repair time distributions. The experimental results revealed that an extended Hedging Corridor policy gave better performance on these two cases. Numerical examples were provided to illustrate contribution of the paper. Zanoni et al. (2014) considered energy reduction in a two-stage fabrication system with controllable fabrication rates, wherein, a single product is first fabricated on an equipment, and then transported in batch shipments to the subsequent fabrication stage. In each stage, the finite fabrication rate is assumed to be adjustable, and this rate is linked to specific energy consumption based on the type of process involved. The purpose of their work was to propose a model in the production planning phase, to analyze the system and minimize its overall costs, including production, inventory, and energy costs. The research result showed that significantly savings were realized as compared to a production plan without considering energy consumption. Additional works (Khouja & Mehrez, 1994; Giri & Dohi, 2005b; Sana, 2010; Liu et al., 2017b; Bottani, et al., 2017; Chiu et al., 2018c,d; Ameen et al., 2018) were also conducted to address various issues and influences of variable fabrication rates on manufacturing systems. As little attention has been paid to study the joint influences of stochastic failures, random scrap, and expedited fabrication rate on the manufacturing runtime decision, this work aims to link the gap.

2. The proposed manufacturing runtime problem

A manufacturing runtime problem with an expedited fabrication rate, random failures, and scrap is explored. Consider a manufacturing system is used to satisfy annual demand rate $\lambda$ of a particular product. The production equipment is subject to stochastic failures which follows Poisson distribution with mean equal to $\beta$ breakdowns per year. An abort/resume (A/R) stock control policy is used when a failure encounters. According to the A/R policy, malfunction equipment is under repair right away, and fabrication of the interrupted/unfinished lot will be resumed when repair task is successfully done. A constant repair time $t_r$ is assumed in this study. To reduce cycle length of the batch fabrication plan, this study adopts an expedited rate. Let $a_1$ represent extra percentage of production rate and consequently, the speedy rate related parameters are defined as follows:
where $P_{1A}$ is the speedy rate, $K_{A}$ and $C_{A}$ are speedy rate relevant setup and unit costs; and $P_{1}$, $K$, $C$, $\alpha_{2}$, and $\alpha_{3}$ denote the standard rate, standard setup and unit costs, and connecting variables between $K_{A}$ and $K$, and between $C_{A}$ and $C$, respectively. For instance, for $\alpha_{1} = 0.3$, it represents that speedy manufacturing rate is 30% higher than standard rate; and $\alpha_{3} = 0.15$ means unit cost is 15% greater than standard unit cost due to the expedited rate. The manufacturing process randomly fabricate $x$ portion of scrap goods at a rate $d_{1A}$ due to diverse unanticipated reasons. Shortages are not permitted, so $(P_{1A} - d_{1A} - \lambda) > 0$ must hold, where $d_{1A}$ is as follows:

$$d_{1A} = xP_{1A}$$

Appendix A includes other notation used in this study. The following two cases must be explored separately because of the random failures:

2.1. Case 1: A random failure taking place

In case 1, time to failure $t < t_{1A}$ and the on-hand stock status in a cycle is illustrated in Fig. 1. It shows that the stock level at $H_0$ when a random failure takes place. After the equipment is repaired and restored, the stock level keeps piling up and arrives at $H$ when manufacturing uptime ends. Then, stock depletes to empty before next cycle starts.

The on-hand status for safety stock is exhibited in Fig. 2. It shows that safety stock being used to satisfy demand during $t_r$, right after a failure taking place. Fig. 3 depicts the on-hand status of scrap in the proposed system with expedited rate, random failures, and scrap. From problem statement and by observing Fig. 1 to Fig. 3, one can obtain the following basic relationships:

$$t_{1A} = \frac{Q}{P_{1A}} = \frac{H}{P_{1A} - d_{1A} - \lambda}$$  \hspace{1cm} (5)

$$t_{2A}' = \frac{H}{\lambda}$$  \hspace{1cm} (6)

$$T_{A}' = t_{1A} + t_r + t_{2A}'$$  \hspace{1cm} (7)

$$H_0 = (P_{1A} - d_{1A} - \lambda)t$$  \hspace{1cm} (8)

$$H = (P_{1A} - d_{1A} - \lambda)t_{1A}$$  \hspace{1cm} (9)

$$d_{1A}t_{1A} = xP_{1A}t_{1A} = xQ$$  \hspace{1cm} (10)
In the case of a random failure taking place, total cost per cycle $TC(t_{1A})$ comprises variable manufacturing, setup, and disposal costs, fixed equipment repair cost, safety stocks’ variable, holding, and delivery costs, and holding costs for perfect and scrap items during manufacturing uptime and depletion time. Hence, $TC(t_{1A})$ is

$$
TC(t_{1A}) = C_sQ + K_A + C_sxQ + M + \left[ C_1\lambda t_r + h_1(\lambda t_r)(t + t_r/2) + C_2\lambda t_r \right] 
+ \frac{H + d_{1A}t_{1A}}{2}(t_{1A}) + (d_{1A}t_r)t_{1A} + \frac{H}{2}(t_{1A})^{\prime}
$$

By applying $E[x]$ to cope with random scrap rate and substituting Eq. (1) to Eq. (10) in Eq. (11), the following $E[TC(t_{1A})]$ can be derived:

$$
E[TC(t_{1A})] = \left[ (1 + \alpha_t)C \right] \left[ (1 + \alpha_t)Pt_{1A} + (1 + \alpha_s)K + C_sE[x] \right] \left[ (1 + \alpha_t)P_{t1A} \right]
+ M + C_1\lambda g + h_1\lambda g \left( t + \frac{E[x]}{2} \right) + C_2\lambda g + h \left[ (1 + \alpha_t)P_t - \lambda \right] t_{1A}
+ \frac{h_1(1 + \alpha_t)Pt_{1A} \gamma}{2} \left[ 1 - E[x] \right] \left( \frac{1 - E[x]}{\lambda} \right) + \frac{2E[x] - 1}{(1 + \alpha_t)P_{t1A}}
$$

The expected cycle time $E[T_{1A}]$ in the case of a random failure taking place, can be derived as follows:

$$
E[T_{1A}] = \frac{Q\left[ 1 - E[x] \right]}{\lambda} = t_{1A} \frac{P_{t1A}}{\lambda} \left[ 1 - E[x] \right]
$$

2.1. Case 2: No random failures taking place

In case 2, time to failure $t > t_{1A}$ and the on-hand stock status in a cycle is illustrated in Fig. 4. It shows that the stock level reaches $H$ when manufacturing uptime finishes. Then, it depletes to empty before next cycle starts.

Fig. 4. On-hand stock status in the proposed system with expedited rate and scrap, but without failures taking place (in green) as compared to that of a manufacturing system with scrap (in black)

Fig. 5. The on-hand status of safety stocks in the proposed system with scrap and an expedited rate, but without failures taking place

Fig. 6. The on-hand status of safety stocks in the proposed system with scrap and expedited rate, but without failures taking place

Fig. 5 displays the on-hand status for safety stock. It shows that safety stocks have not been used because no machine failures take place. Fig. 6 illustrates the on-hand status of scrap in the proposed system with expedited rate and scrap, but without failures taking place. From problem statement and also by observing Fig. 4 to Fig. 6, one can obtain the following basic relationships:

$$
t_{1A} = \frac{Q}{P_{t1A}} = \frac{H}{P_{t1A} - d_{1A} - \lambda}
$$

$$
t_{2A} = \frac{H}{\lambda}
$$
\[ H = (P_{1A} - d_{1A} - \lambda) t_{1A} \]  
\[ T_A = t_{1A} + t_{2A} \]  

In the case of no machine failures, total cost per cycle \( TC(t_{1A}) \) comprises variable manufacturing, setup, and disposal costs, safety stocks' holding cost, and holding costs for perfect and scrap items during manufacturing uptime and depletion time. Hence, \( TC(t_{1A}) \) is

\[ TC(t_{1A}) = C_A Q + K_A + C_S + \lambda t_f + h_t (\lambda t_f) T_A + h \left( \frac{H + d_{1A} t_{1A}}{2} + \frac{H}{2} t_{2A} \right) \]  

By applying \( E[x] \) to handle with random scrap rate and substituting Eqs. (1-4) and Eqs. (14-17) in Eq. (18), the following \( E[TC(t_{1A})] \) can be derived:

\[ E[TC(t_{1A})] = \left( (1 + \alpha_3) C \right) P_{1A} + \left( (1 + \alpha_2) K + C_S E[x] \right) P_{1A} \]
\[ + h_t \lambda g T_A + \frac{h_t \left( (1 + \alpha_3) C \right)}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x]}{\alpha} \right] \]  

3. Solving the proposed runtime problem

Because equipment failure is assumed to be a random variable which follows the Poisson distributed with mean equal to \( \beta \) per year, hence, time-to-failure \( t \) obeys an Exponential distribution with \( f(t) = \beta e^{-\beta t} \) and \( F(t) = (1 - e^{-\beta t}) \) as its density and cumulative density functions, respectively. Therefore, the expected annual system cost \( E[TCU(t_{1A})] \) can be determined as follows:

\[ E[TCU(t_{1A})] = \frac{\int_{t_{1A}}^{T_A} E[TC(t_{1A})] \cdot f(t) \, dt + \int_{T_A}^{\infty} E[TC(t_{1A})] \cdot f(t) \, dt}{E[T_A]} \]  
\[ \text{where } E[T_A] = \int_{0}^{t_{1A}} E[T_A] \cdot f(t) \, dt + \int_{t_{1A}}^{\infty} E[T_A] \cdot f(t) \, dt = \frac{t_{1A} \left( (1 + \alpha_1) P \right) \left[ 1 - E[x] \right]}{\lambda} \]  

The following \( E[TCU(t_{1A})] \) can be gained by substituting Eq. (12), Eq. (19), and Eq. (21) in Eq. (20) (for detailed derivations please see Appendix B):

\[ E[TCU(t_{1A})] = \frac{\lambda}{\left( 1 - E[x] \right)} \left[ \frac{Z_t + Z_s + Z_s e^{-\beta t_{1A}} + Z_s e^{-\beta t} + \left( (1 + \alpha_3) C \right) + C_s E[x]}{t_{1A}} \right] \]
\[ + \frac{h_t \left( (1 + \alpha_3) C \right)}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x]}{\alpha} \right] \]
\[ + h_t g \left( 1 - E[x] \right) \left( e^{-\beta t_{1A}} \right) \]  

3.1 Convexity of \( E[TCU(t_{1A})] \)

We first apply the first- and second-derivatives of \( E[TCU(t_{1A})] \) and gain the following:

\[ \frac{dE[TCU(t_{1A})]}{dt_{1A}} = \frac{\lambda}{\left( 1 - E[x] \right)} + \frac{h_t \left( (1 + \alpha_3) P \right)}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x]}{\alpha} \right] \]
\[ - \beta h_t g \left( 1 - E[x] \right) \left( e^{-\beta t_{1A}} \right) \]
From Eq. (24), because the first term \( \lambda / (1 - E[x]) \) on RHS (right-hand side) is positive, it follows that \( E[TCU(t_{1A})] \) is convex if the second term on RHS of Eq. (24) is also positive. That is if Eq. (25) holds.

\[
\gamma(t_{1A}) = \frac{2(Z_1 + Z_2 + Z_4 e^{-\beta_{1A}})}{(-t_{1A}^2 \beta Z_2 e^{-\beta_{1A}} - t_{1A}^2 \beta^2 h g (1 - E[x]) (e^{-\beta_{1A}}) - t_{1A} \beta^2 Z_4 e^{-\beta_{1A}} - 2 \beta Z_4 e^{-\beta_{1A}})} > t_{1A} > 0
\]  

3.2 Searching for \( t_{1A}^* \)

We set the first-derivative of \( E[TCU(t_{1A})] \) equal to zero to search for optimal runtime \( t_{1A}^* \) under the condition that Eq. (25) holds.

\[
\lambda \left[ 1 - E[x] \right] \left\{ \frac{-Z_2}{t_{1A}^3} - \beta Z_2 e^{-\beta_{1A}} - \frac{Z_4 e^{-\beta_{1A}}}{t_{1A}^3} - \frac{Z_4 e^{-\beta_{1A}}}{t_{1A}} - \beta h g (1 - E[x]) (e^{-\beta_{1A}}) \right\} + \frac{h(1 + \alpha_i) P_t}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_i) P_t} \right] = 0
\]

or

\[
\left\{ \frac{-\beta Z_2 e^{-\beta_{1A}} - \beta h g (1 - E[x]) (e^{-\beta_{1A}})}{t_{1A}^3} + \frac{h(1 + \alpha_i) P_t}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_i) P_t} \right] \right\} + t_{1A} \left( -\beta Z_4 e^{-\beta_{1A}} + \left( -Z_1 - Z_2 - Z_4 e^{-\beta_{1A}} \right) \right) = 0
\]

Let \( r_2, r_1, \) and \( r_0 \) stand for the following:

\[
r_2 = \left\{ -\beta Z_2 e^{-\beta_{1A}} - \beta h g (1 - E[x]) (e^{-\beta_{1A}}) + \frac{h(1 + \alpha_i) P_t}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_i) P_t} \right] \right\} ;
\]

\[
r_1 = -\beta Z_4 e^{-\beta_{1A}} ;
\]

\[
r_0 = -(Z_1 + Z_2 + Z_4 e^{-\beta_{1A}})
\]

Eq. (28) becomes

\[
r_2 \left( t_{1A} \right)^2 + r_1 \left( t_{1A} \right) + r_0 = 0
\]

We can now apply Eq. (30) (i.e., the square roots solution procedure) to search for \( t_{1A}^* \):
\[ t_{1A}^* = \frac{-r_0 \pm \sqrt{r_0^2 - 4r_0 r_b}}{2r_2} \] (30)

\[ Z_0 e^{\beta h_{1A}} \pm \left( Z_0 e^{\beta h_{1A}} \right)^2 - 4 - \beta Z_0 e^{\beta h_{1A}} - \beta h_{1A} g \left( 1 - E[x] \right) \left( e^{\beta h_{1A}} \right) + \frac{h(1 + \alpha_1)P_1}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_1)P_1} \right] \left( -Z_1 - Z_2 - Z_0 e^{\beta h_{1A}} \right) \] (31)

\[ t_{1A}^* = \frac{h(1 + \alpha_1)P_1}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_1)P_1} \right] - Z_1 - Z_2 \]

3.2.1 Recursive algorithm for locating \( t_{1A}^* \)

The cumulative density function of Exponential distribution \( F(t_{1A}) = (1 - e^{-t_{1A}}) \) and it is over the interval of \([0, 1]\), so does its complement \( e^{-t_{1A}} \). Furthermore, one can rearrange Eq. (27) as follows:

\[ e^{-t_{1A}} = \frac{h(1 + \alpha_1)P_1}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_1)P_1} \right] - Z_1 - Z_2 \]

The proposed recursive algorithm is as follows: Step (1): first set \( e^{-t_{1A}} = 0 \) and \( e^{-t_{1A}} = 1 \), use Eq. (32) to compute the upper and lower bounds for uptime \( t_{1A} \) (i.e., \( t_{1A}^U \) and \( t_{1A}^L \)). Step (2): use current \( t_{1A}^U \) and \( t_{1A}^L \) to calculate and update \( e^{-t_{1A}^U} \) and \( e^{-t_{1A}^L} \). Step (3): use current \( e^{-t_{1A}^U} \) and \( e^{-t_{1A}^L} \) to compute Eq. (29) again and update \( t_{1A}^U \) and \( t_{1A}^L \) for uptime \( t_{1A} \). Step (4): verify whether or not \( (t_{1A}^U - t_{1A}^L) = 0 \), if it is true, then go to Step (5); otherwise, go to Step (2). Step (5): \( t_{1A}^* \) is located (it is either \( t_{1A}^U \) or \( t_{1A}^L \)).

4. Numerical demonstration

To demonstrate applicability of the proposed manufacturing runtime problem, an example with the following assumption of system parameters (see Table 1) is considered:

Table 1

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Assumptions of system parameters in the numerical demonstration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>4000 * 20% * 0.5 * 15000 * 10000 * 0.1 * 495 * 450 * 0.25 * 1</td>
</tr>
<tr>
<td>Parameters</td>
<td>( C_A ) * ( C ) * ( C_K ) * ( M ) * ( g ) * ( h ) * ( h_3 ) * ( C_I ) * ( C_I )</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>2.5 * 2.0 * 0.3 * 2500 * 0.018 * 0.8 * 0.8 * 0.01 * 2.0</td>
</tr>
</tbody>
</table>

Prior to solving \( t_{1A}^* \) for the problem, the convexity of \( E[T_{CU}(t_{1A})] \) must be verified first. That is, to test whether \( \gamma(t_{1A}) > t_{1A} > 0 \) (i.e., Eq. (25)). We start with setting \( e^{-t_{1A}} = 0 \) and \( e^{-t_{1A}} = 1 \), then applying Eq. (31) to obtain \( t_{1A}^U = 0.5000 \) and \( t_{1A}^L = 0.1130 \). Finally, by computing Eq. (25) using these \( t_{1A}^U \) and \( t_{1A}^L \) values, we confirmed \( \gamma(t_{1A}) = 0.7466 > t_{1A} > 0 \) and \( \gamma(t_{1A}) = 0.2966 > t_{1A} > 0 \). So, the convexity of \( E[T_{CU}(t_{1A})] \) is proved. Furthermore, to demonstrate the proposed study is applicable for broader range of mean machine failure rates, extra convexity tests were carried out and results are exhibited in Table C-1 (Appendix C). A recursive algorithm (see subsection 3.2.1) is utilized to locate \( t_{1A}^* \). The detailed solution processes are shown in Table 2, where \( t_{1A}^* = 0.2015 \) and \( E[T_{CU}(t_{1A}^*)] = $13,536 \) are obtained.

The impact of differences in \( t_{1A} \) on diverse cost elements in \( E[T_{CU}(t_{1A})] \) are depicted in Fig. 7. It indicates that as uptime \( t_{1A} \) deviates from its optimal position (i.e., 0.2015), \( E[T_{CU}(t_{1A})] \) begins to boost; and as \( t_{1A} \) increases, holding cost goes up significantly and quality relevant cost raises accordingly; but the setup cost declines drastically. Fig. 8 exhibits the effect of variations in random scrap rate \( x \) on optimal
manufacturing uptime $t_{1A}^*$. It shows that optimal uptime $t_{1A}^*$ increases notably, as scrap rate $x$ raises; and at $x = 0.2$ (as assumed in the example), $t_{1A}^* = 0.2015$ years. The influence of variations in mean-time-to-failure $1/\beta$ together with various defective rates $x$ on $E[TCU(t_{1A}^*)]$ is illustrated in Fig. 9.

**Table 2**

<table>
<thead>
<tr>
<th>Step no.</th>
<th>$t_{1AU}$</th>
<th>$e^{-\beta t_{1AU}}$</th>
<th>$t_{1AL}$</th>
<th>$e^{-\beta t_{1AL}}$</th>
<th>Difference between $t_{1AU}$ and $t_{1AL}$</th>
<th>$E[TCU(t_{1AU})]$</th>
<th>$E[TCU(t_{1AL})]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>0.5000</td>
<td>-</td>
<td>0.1130</td>
<td>-</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0.2887</td>
<td>0.7493</td>
<td>0.1669</td>
<td>0.8463</td>
<td>0.1285</td>
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<td>$13,562.57$</td>
</tr>
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<td>0.1886</td>
<td>0.8281</td>
<td>0.0424</td>
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<td>$13,539.65$</td>
</tr>
<tr>
<td>3</td>
<td>0.2119</td>
<td>0.8090</td>
<td>0.1968</td>
<td>0.8214</td>
<td>0.0151</td>
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<td>$13,536.84$</td>
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<td>$13,536.43$</td>
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<tr>
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<td>0.2015</td>
<td>0.8175</td>
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<td>$13,536.43$</td>
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<td>0.2015</td>
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<td>0.0000</td>
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<td>$13,536.43$</td>
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</tbody>
</table>

It specifies that as $1/\beta$ increases (which implies the occurrence of a machine failure is less likely), $E[TCU(t_{1A}^*)]$ declines significantly, especially when $1/\beta \geq 0.25$. It also shows that as scrap rate $x$ goes up, $E[TCU(t_{1A}^*)]$ increases notably.

Analytical results on cost elements in $E[TCU(t_{1A}^*)]$ is exhibited in Fig. 10. It reveals that 16.9% of $E[TCU(t_{1A}^*)]$ is related to the expedited rate option, 8.1% of system cost is concerned with product quality matters, and 5.6% is regarding the machine failure matter, etc. The impact of changes in expedited ratio $P_{1A}/P_1$ on $E[TCU(t_{1A}^*)]$ is shown in Fig. 11. It indicates that as $P_{1A}/P_1$ increases, $E[TCU(t_{1A}^*)]$ goes up significantly; and it reconfirms that when $P_{1A}/P_1 = 1.5$, $E[TCU(t_{1A}^*)] = $13,536 (as assumed in our example).
Fig. 12 displays the combined effect of differences in factor of expedited rate $\alpha_1$ and mean-time-to-failure $1/\beta$ on $t_{1A^*}$. It points out that as $1/\beta$ raises, $t_{1A^*}$ decreases considerably; and as the factor of expedited rate $\alpha_1$ increases, the optimal uptime $t_{1A^*}$ decreases significantly (this reconfirms that manufacturing uptime reduced extensively as expedited rate boosts). Fig. 13 illustrates the influence of variations in expedited rate $P_{1A}/P_1$ on utilization (i.e., $t_{1A}/E[T_A]$). It shows that as $P_{1A}/P_1$ increases, machine utilization decreases notably; and at $P_{1A}/P_1 = 1.5$ (as assumed in our example), utilization declines to 0.2963 from 0.4444 (see Table 3 for details).

Fig. 14 exhibits the impact of changes in random scrap rate $x$ on different cost elements in $E[TCU(t_{1A^*})]$. It specifies that as $x$ increases, product quality cost boosts extensively.

![Fig. 12. Combined effect of differences in expedited rate factor $\alpha_1$ and mean-time-to-failure $1/\beta$ on $t_{1A^*}$](image)

![Fig. 13. The influence of variations in expedited ratio $P_{1A}/P_1$ on machine utilization](image)

![Fig. 14. The impact of variations in random scrap rate $x$ on different cost contributors of $E[TCU(t_{1A^*})]$](image)

![Fig. 15. Combined influence of variations in $1/\beta$ and factor of expedited rate $\alpha_1$ on $E[TCU(t_{1A^*})]$](image)

<table>
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<tr>
<th>$P_{1A}/P_1$</th>
<th>$t_{1A}$</th>
<th>Increase %</th>
<th>Utilization $(t_{1A}/T_A)$</th>
<th>Decline %</th>
<th>$E[TCU(T_A^*)]$</th>
<th>Increase %</th>
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<td>0.4444</td>
<td>-</td>
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<td>-</td>
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<td>-9.09%</td>
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<td>3.71%</td>
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<tr>
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<tr>
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<td>-37.50%</td>
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<td>53.68%</td>
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<tr>
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<td><strong>0.1709</strong></td>
<td><strong>-61.54%</strong></td>
<td><strong>$18,411$</strong></td>
<td><strong>61.54%</strong></td>
</tr>
<tr>
<td>2.70</td>
<td>0.1109</td>
<td>-67.78%</td>
<td>0.1646</td>
<td>-62.96%</td>
<td>$18,860$</td>
<td>65.48%</td>
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</tbody>
</table>
Combined influence of variations in mean-time-to-failure $1/\beta$ and the factor of expedited rate $\alpha_1$ on $E[T(\text{TCU}(t_{1A}^{*}))]$ is depicted in Fig. 15. It indicates that as the mean-time-to-failure $1/\beta$ increases, $E[T(\text{TCU}(t_{1A}^{*}))]$ declines noticeably; and as the factor of expedited rate $\alpha_1$ increases, $E[T(\text{TCU}(t_{1A}^{*}))]$ boosts drastically. Fig. 16 illustrates the joint effect of differences in manufacturing uptime $t_{1A}$ and random scrap rate $x$ on $E[T(\text{TCU}(t_{1A}^{*}))]$. It points out the effect of $t_{1A}$ and convexity on cost function, and as random scrap rate $x$ goes up, $E[T(\text{TCU}(t_{1A}^{*}))]$ increases radically.

Fig. 16. The joint effect of differences in uptime $t_{1A}$ and random scrap rate $x$ on $E[T(\text{TCU}(t_{1A}^{*}))]$

Fig. 17. Behavior of “system cost increase percentage” versus “machine utilization decline percentage”

Fig. 17 demonstrates the behavior of “system cost increase percentage” versus “machine utilization decline percentage.” It reveals that the linear breakeven point of cost/benefit is at $P_{1A}/P_1 = 2.60$, where the percentage of cost increase is equal to the percentage of utilization decline (refer to Table 3 for details).

5. Concluding Remarks

This study explores the optimal runtime solution for a manufacturing system featuring an expedited fabrication rate, random equipment failures, and scrap. In order to explicitly represent realistic features of the studied problem, an accurate model comprising two separate situations is constructed (refer to the cases 1 and 2 in subsections 2.1 and 2.2, respectively.) Mathematical derivation and optimization approaches (including a proposed algorithm) are used to find the system cost and optimal runtime solution for the problem (see section 3). Finally, the applicability and sensitivity analyses of research outcome are illustrated via a numerical example (refer to section 4).

In addition to deriving the optimal runtime policy for the problem, the major contribution of the present work includes to reveal diverse critical information regarding the individual/joint influences of variations in the expedited rate, random scrap rate, and stochastic time-to-failure on the optimal runtime decision (see Figures 8 and 12), on total system expenses (refer to Figures 7, 9, 11, 15, and 16), on different cost components (see Figures 10 and 14), and on machine utilization (see Figures 13 and 17). In summary, the research outcomes not only enable the in-depth problem analyses, but also facilitate managerial decision makings. For future study, consideration of a random demand in the context of the same problem will be an interesting direction.

Acknowledgment

Authors express their appreciation to the Ministry of Science and Technology of Taiwan for sponsoring this work (under funding#: MOST 107-2221-E-324-015).
References


Management Sciences, 38, 104-123.

**Appendix – A**

The following are other notation used by this study:

- $t$ = manufacturing time (in years) before a random failure taking place,
- $M$ = fixed cost to repair the machine failure,
- $Q$ = lot size,
- $T'_{A}$ = cycle time of the studied system,
- $E[x]$ = the expected random scrap rate,
- $H_{0}$ = level of perfect stock when a random failure takes place,
- $H$ = level of perfect stock when manufacturing uptime ends,
- $h$ = unit holding cost,
- $C_{1}$ = unit cost for safety stock,
- $h_{3}$ = unit holding cost for safety stock,
- $t_{1A}$ = uptime – the decision variable of the proposed manufacturing runtime problem,
- $t'_{2A}$ = stock depletion time of the proposed system with a random failure taking place,
- $g$ = fixed machine repair time, thus, $g = t_{r},$
- $I(t)$ = on-hand stock status at time $t,$
- $I_{s}(t)$ = on-hand status of scraps at time $t,$
- $I_{r}(t)$ = on-hand status of safety stocks at time $t,$
- $E[T'_{A}]$ = expected cycle length in the case of a random failure taking place,
- $TC(t_{1A})_{1}$ = total cost per cycle in the case of a random failure taking place,
- $E[TC(t_{1A})]_{1}$ = expected total cost per cycle in the case of a random failure taking place,
- $t_{2A}$ = stock depletion time in the case of no machine failure taking place,
- $T_{A}$ = cycle time in the case of no machine failure taking place,
- $E[T_{A}]$ = expected cycle time in the case of no machine failure taking place,
- $TC(t_{1A})_{2}$ = total cost per cycle in the case of no machine failure taking place,
- $E[TC(t_{1A})]_{2}$ = expected total cost per cycle in the case of no machine failure taking place,
- $t_{1}$ = uptime for a manufacturing system with scrap only,
- $t_{2}$ = stock depletion time for a manufacturing system with scrap,
- $T_{A}$ = cycle time for a manufacturing system with scrap,
- $E[T_{A}]$ = expected cycle time for the proposed system with expedited rate, random failures, and scrap,
- $E[TCU(t_{1A})]$ = expected annual system cost for the proposed system with expedited rate, random failures, and scrap.

**Appendix – B**

The following are detailed derivations for $E[TCU(t_{1A})]$ (i.e., Eq. (20)).

Let $u_{1}$ and $u_{2}$ be the following:

$$u_{1} = \left[ (1 + \alpha_{i}) C \right] P_{i} + C_{s} E[x] (1 + \alpha_{i}) P_{i}$$  \hspace{1cm} (B-1)

$$u_{2} = \frac{h}{2} \left[ (1 + \alpha_{i}) P_{i} \right]^{2} \left[ 1 - E[x] \right]^{2} \frac{2 \left[ 2 E[x] - 1 \right]}{\lambda (1 + \alpha_{i}) P_{i}}$$  \hspace{1cm} (B-2)

then $E[TC(t_{1A})]_{1}$ (Eq. (10)) and $E[TC(t_{1A})]_{2}$ (Eq. (17)) become the following:
Applying the following Eq. (18) and substituting Eqs. (B-3), (B-4) and (19) in Eq. (18), Eq. (B-5) can be gained after extra derivation efforts.

\[
E\left[TC(t_{1A})\right] = \frac{\int_{0}^{t_{1A}} E\left[TC(t_{1A})\right] \cdot f(t)dt + \int_{t_{1A}}^{\infty} E\left[TC(t_{1A})\right] \cdot f(t)dt}{E[T_{1A}]} 
\]

\[
E\left[TCU(t_{1A})\right] = h_{1A} g \left( e^{-\rho_{1A}} \right) \frac{\left[\left(1 + \alpha_{1}\right)K\right]}{(1 + \alpha_{1})P_{1A}} + \frac{[\alpha_{1}]C_{1} E[x] + t_{1A} \left\{ h\left(1 + \alpha_{1}\right)P_{1} \frac{\left[1 - E[x]\right]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_{1})P_{1}\beta} \right\} + \frac{1}{t_{1A}} + \frac{\lambda}{1 - E[x]} + e^{-\rho_{1A}} \left[ -hg - \frac{h_{1A} g}{(1 + \alpha_{1})P_{1}} + \frac{h\lambda g}{(1 + \alpha_{1})P_{1}} \right] + e^{-\rho_{1A}} \left[ -h_{1A} g - \frac{M}{(1 + \alpha_{1})P_{1}} - \frac{h_{1A} g^{2}}{2(1 + \alpha_{1})P_{1}} - \frac{h_{1A} g}{\beta(1 + \alpha_{1})P_{1}\beta} \right]}{(1 + \alpha_{1})P_{1}\beta} 
\]

Suppose we let \(Z_{1}, Z_{2}, Z_{3}, Z_{4}\) denote the following:

\[
Z_{1} = \left[\frac{(1 + \alpha_{1})K}{(1 + \alpha_{1})P_{1}}\right] \quad (B-6)
\]

\[
Z_{2} = \left[\frac{C_{1}\lambda g}{(1 + \alpha_{1})P_{1}} + \frac{C_{1}\lambda g}{(1 + \alpha_{1})P_{1}} + \frac{M}{(1 + \alpha_{1})P_{1}} + \frac{h_{1A} g^{2}}{2(1 + \alpha_{1})P_{1}} + \frac{h_{1A} g}{\beta(1 + \alpha_{1})P_{1}\beta} - \frac{h\lambda g}{(1 + \alpha_{1})P_{1}\beta}\right] \quad (B-7)
\]

\[
Z_{3} = \left[-hg - \frac{h_{1A} g}{(1 + \alpha_{1})P_{1}} + \frac{h\lambda g}{(1 + \alpha_{1})P_{1}}\right] \quad (B-8)
\]

\[
Z_{4} = \left[\frac{C_{1}\lambda g}{(1 + \alpha_{1})P_{1}} + \frac{C_{1}\lambda g}{(1 + \alpha_{1})P_{1}} + \frac{M}{(1 + \alpha_{1})P_{1}} + \frac{h_{1A} g^{2}}{2(1 + \alpha_{1})P_{1}} - \frac{h_{1A} g}{\beta(1 + \alpha_{1})P_{1}\beta} + \frac{h\lambda g}{(1 + \alpha_{1})P_{1}\beta}\right] \quad (B-9)
\]

then
\[ E[TCU(t_{1A})] = \lambda \left( \frac{Z_t + Z_{1A} + Z_{2t}e^{-\beta_{tA}}}{t_{1A}} + \frac{Z_{2t}e^{-\beta_{tA}}}{t_{1A}} + \left[(1 + \alpha_t)C + C_tE[x]\right] \right) \]

\[ + h_{tA} \left[ \frac{h(1 + \alpha_t)P_t}{2} \left[ \frac{1 - E[x]}{\lambda} + \frac{2E[x] - 1}{(1 + \alpha_t)P_t} \right] \right] \]

\[ + h_{tA} \left[ 1 - E[x] \right] \left( e^{-\beta_{tA}} \right) \]

(20)

Appendix – C

Table C-1
Results of extra convexity tests with broader range of \( \beta \) values

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<th>( \beta )</th>
<th>( t_{1AU} )</th>
<th>( \gamma(t_{1AU}) )</th>
<th>( t_{1AL} )</th>
<th>( \gamma(t_{1AL}) )</th>
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