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Introduce free replacement extended warranty and bundle it? Optimal new extended warranty introduction strategy

Kaiying Cao[a] and Yunyi Su[b*]

[a]College of International Tourism and Public Administration, Haihan University, Haikou 570228, PR China
[b]School of Economics and Management, Tongji University, 800 Caoan Road, 201804 Shanghai, China

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A B S T R A C T

To meet consumer replacement needs, the free replacement extended warranty (FREW) is born and becomes popular in the extended warranty (EW) market. In this context, firms need to consider whether to introduce the FREW. Given the limited resources of the firms and cannibalism caused by the FREW, firms need to decide how to introduce the FREW. To address these issues, we construct theoretical models and obtain some managerial insights. We find that the optimal introduction strategy is related to the development cost and the expansion effect on the product market. Moreover, the optimal bundling strategy is affected by the unit maintenance cost and the cost discount caused by the FREW. Only when the benefit of the FREW is great enough, is bundling always better. An interesting result is that the price of the bundled EW is higher than the sum of the EWs' prices when selling EWs separately.

1. Introduction

The extended warranty (hereafter EW), an indispensable part of the product and service strategies of the modern firm due to its profitability and the effect on promoting product sales (Chen, 2019; Lou *et al.*, 2019), has developed rapidly and has been diverse. There exists a variety of new extended warranties, such as free replacement, accident protection plan, upgrade replacement, and so on. Those are different from the traditional extended warranty (TEW), which offers consumers who purchase it a free maintenance service for the damaged product during the EW coverage period. Especially the free replacement EW (FREW) provides a totally new product to replace the damaged product, which has great appeal to the consumers. It has also become a widely used service that is provided for a huge number of products. For instance, Amazon.com has provided the free replacement for its kindle (Amazon.com, 2022), and Sony has offered this service for some headset products (Sony.com, 2022). It is worth considering the value of providing the FREW for firms. Thus, we focus on this type of new EW in this paper. Except for the extensive EW service range, the sales method of EW is varied. The EW package, including two or more EWs, has become more popular among these methods. For example, Apple has launched its EW package, that is, AppleCare+ (Apple.com, 2022). Samsung offers Samsung Care+ for its products (Samsung.com, 2022). JD.com and Amazon.com have provided similar EW packages (Amazon.com, 2022; JD.com, 2022). Despite the popularity of the bundling method when selling the EW, it is still unclear whether this method will benefit firms or not. Nowadays, the great majority of products have been equipped with the TEW. However, whether firms provide a new EW should still be scrutinized meticulously due to the firm's limited resources and high development cost. If the firm decides to introduce a new EW, such as the FREW, while benefiting from the attractiveness of this unique service and the improvement in product demand, the expensive development cost should be borne by the firm. But if the firm chooses to maintain the status quo, it may miss the opportunities to make more profits and lose its competition. So, the firm should trade off the benefits and costs of providing a new EW (i.e., FREW). Is it an intelligent choice of the firm to change its EW strategy, that is, offer the FREW? This is one of the critical issues that our paper wants to explore.

* Corresponding author
E-mail suemuchen@163.com (Y. Su)
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And then, if the firm decides to introduce FREW, the sales method of this service has become a significant problem. When selling the new service separately, the new EW's encroachment may hurt the TEW's demand, but it will attract some consumers who are interested in the new EW only. Conversely, if the new service bundles with the TEW and is sold as an EW package, the higher price of the package may decrease the service demand. Besides, the firm will lose some customers who only want to purchase the TEW or the FREW. Therefore, both choices have their own strong and weak points. It is a challenge for firms to decide whether to bundle the FREW with the TEW. To address the issues and challenges mentioned above, we consider a firm and construct some theoretical models. Due to the attractiveness and popularity of the FREW, it is necessary for the firm to explore whether to introduce the FREW. Then, according to the issues of whether to raise the FREW and the challenge of which sales method is better---bundling sales or selling extended warranties separately, we propose three cases: the first is that the firm sells the product and provides the TEW only (O); the second is that the firm sells the product with two kinds of the EWs sold separately (TS); the last is that the firm sells the product and provides a new EW package including the TEW and the FREW (TB). After calculating these models and presenting the optimal decisions, we compare the optimal solutions and profits and give some managerial insights. In the extension, we take into account corporate social responsibility and the two-tier supply chain.

The rest of the paper is organized as follows. We review and summarize some relevant literature in the Literature review Section. Section 3 describes the issues we want to explore in this paper, introduces the variable setting, the demand function, and the cost structure, and proposes the firm's profit function. The equilibrium solutions are presented in Section 4, and we analyze these decisions and profits in this section. Section 5 shows the extension model we constructed, with the corporate social responsibility and the supply chain being considered. The effect of these factors on the EW strategy is also presented. Finally, we conclude and give some managerial insights in Section 6. Note that all the proof processes are placed in the Appendix.

2. Literature review

This paper explores the introduction and bundling strategies of new EW (i.e., FREW). Thus, we review the three most related literature on the EW, the free replacement service, and the bundling sale. The first stream studies the EW. Due to the considerable profit (Kelley and Conant, 1991; Soberman, 2003), incentive function (Dybvig and Lutz, 1993; Gallego *et al.*, 2014; Cao and He, 2018), the signal function of the EW (Li *et al.*, 2019; Lu and Shang, 2019), there is already widespread research on the EWs. Some of them focus on the pricing problem of the EW (Hartman and Laksana, 2009; Zheng *et al.*, 2021) and some of them explore who is the best provider and the best seller of it (Desai and Padmanabhan, 2004; Li *et al.*, 2012; Mai *et al.*, 2017). The interaction effect of the base warranty and EW is also investigated. Jiang and Hsiao (2011) point out that the retailer's EW has a negative impact on the manufacturer's warranty if the public can assess the product quality and the manufacturer offers the base warranty only when it is cost-efficient in providing such a service. The literature above is based on the one-dimension EW. There are an increasing number of papers on the two-dimension EW, such as Wang *et al.* (2017), and Wang and Ye (2020). Most of the above papers investigate the traditional EW, providing free repair service for damaged products. According to these literature reviews, it is evident that the area of traditional EW has been deeply explored. However, the area of the new EW is rarely studied and has limited research. Among this limited research, the trade-in service as a new EW has been investigated. For example, Bian *et al.* (2019) explore the optimal EW strategy considering the additional trade-in service as the new EW. The results reveal that whether to offer the trade-in EW is related to the handling cost for the used product, and the new EW will be sold at a lower price. The other new EWs, such as accident protection insurance, FREW, and so on, have almost nobody to explore. Although Cao *et al.* (2022a) compare the traditional EW and the FREW, the coexistence of the two EWs in the market has not been considered. Thus, this paper intends to fill this gap.

The second stream is the free replacement service strategy. Although there are few literature studies on it as an EW, some scholars have explored it as a warranty strategy (Rao, 2021; Qiao *et al.*, 2022). For instance, Tsoukalas and Agrafiotis (2013) propose a new warranty policy, the free replacement, for non-repairable products characterized by age and usage. The total warranty cost and its expected value are obtained to give some advice for the compensation policy and the evaluation of the policy's performance. There exist relatively affluent papers that analyze the cost of the free replacement warranty (Liu *et al.*, 2015; Wang *et al.*, 2019). Also, with the free replacement policy, the optimal replacement policy is investigated by some scholars. Chien (2008) derives the optimal replacement age considering the renewing free-replacement warranty policy, Chang et al. (2019) provide a multi-parameter preventative replacement strategy for a stochastically deteriorating system and Liu *et al.* (2021) examine that the optimal strategy is unique even if the product has various failure modes and is equipped with a rebate warranty and the free replacement warranty. Moreover, more and more scholars are concerned about the effect of the free replacement warranty on the operational decisions of the supply chain recently. For example, Wu *et al.* (2009) obtain the optimal price, warranty length, and production rate of the product in the static demand market considering the existence of the free replacement warranty and conclude that the inventory volume will affect the operation strategy. Liu *et al.* (2020) relate the remanufactured products to the non-renewing free replacement warranty and explore the optimal pricing and production strategy of the new and remanufactured products in the competitive market. Vafaeinejad and Sajadieh (2022) combine trade-in with free replacement warranty considering strategic customers and then propose a heuristic algorithm to search concurrent optimization. Despite the increasing number of papers exploring the impact of free replacement on operation management, there is still a gap in the FREW, much less in the operational decisions of the supply chain with the FREW. This paper aims to study FREW and its operational decisions, such as pricing and sales methods. It's imperative to delve into the new EW and provide some insights for the firms.The last stream is the bundling problem. The bundling strategy can be divided into pure and mixed bundling (Derdenger and Kumar, 2013; Chen *et al.*, 2020). The pure bundling strategy was widely studied several years ago (Prasad *et al.*, 2010; Girju *et al.*, 2013). Recently, there has been increasing interest in the latter one. Bhargava (2013) compares the partially mixed bundling strategy with the full one when the products are independent and concludes that the difference in the product valuation affects the optimal bundling strategy. Besides, Shao and Li (2019) investigate complementary products' partial mixed bundling strategy. They find that the retailer prefers a bundling strategy in the context of channel competition. Guo *et al.* (2021) discuss the optimal bundling strategy for the independent products of the retailer platform, with mixed bundling being considered. The above literature focuses on the tangible product. The bundling problem of information products is also examined by some scholars, such as Zhang *et al.* (2016), Banciu *et al.* (2022), and Cao *et al.* (2022b). However, the product relationships in all the research are independent or complementary, and the bundling problem of substituted products is rarely studied. Our paper studies the bundling problem of the TEW and the FREW, and a substitution relation exists between them. Unlike Chen *et al.* (2021), who consider the presence of substitutability and complementarity of the tangible product and analyze the optimal bundling strategy in a distribution channel, we focus on the bundling problem of service products and incorporate the EW introduction strategy into the bundling strategy. Also, we differ with Yuan and Xiao (2022) that explores the bundling strategy of high-end product and low-end product and take the interaction of the pricing strategy of the product and EWs into account. Although Zhang and Gao (2021) investigate the bundling problem of the product and the EW, our work differs from it in considering the bundling problem of various EWs.

In conclusion, our paper studies the new EW introduction and bundling problem considering the substitutability of the FREW and the TEW, which is different from previous research, and expands the research fields of EW and bundling problems. Some insights we obtained may contribute to operation management.

3. Problem description and model setup

We consider a firm that sells a single product and offers a traditional EW. The occurrence of failure of this product follows a nonhomogeneous Poisson process with $\Lambda(T)$ as the cumulative failure rate of the product. It also means the expected number of failures during the EW period $[0,T]$, which is given as $\Lambda(t)=\int_0^T \lambda(t)dt$. Note that, $\lambda(t)$ denote the failure rate of the product at time t. Following Jung, Park, and Park (2015) and Liu *et al.* (2020), we assume that $\lambda(t)$ is a constant, that is, $\lambda(t)=\upsilon$, which means the failure time for the product follows the exponential distribution with rate υ. The traditional EW is a charged service that provides free maintenance service for the product during the EW period. Now, to attract consumers who are dissatisfied with the existing EW and improve its competitive edges, the firm considers introducing a new EW, i.e., free replacement EW (FREW), that offers limited times free replacement services for the destroyed product during the EW period T. The free replacement chances the firm provided is N. Consumers cannot replace their damaged product freely if they use up all the replacement chances or the product's damaged time point is beyond the EW period. However, the sale approach of the FREW---separate sale or bundle sale, should be deliberated. If the firm sells this service separately, consumers interested in EWs can buy the TEW, the FREW, or both. However, if the firm sells FREW bundling with the TEW, consumers can only choose to buy the TEW or the bundled product, including the TEW and the FREW. Then, according to the firm's EW strategy, we consider three cases: the firm only provides the TEW (O); the firm introduces the FREW and sells the TEW and the FREW separately (TS); and the firm introduces the FREW and sells it bundled with the TEW (TB). Note that we use superscripts O, TS, and TB to represent these three cases O, TS, and TB. Moreover, the demand functions are derived in Section 3.1, and the cost structures are presented in Section 3.2. Finally, Section 3.3 presents the firm's profit models in different cases.
To clarify our model, we present some notations in Table 1.

Table 1
The model notation

Notation	Description
k	The consumer's perceived value of the per unit EW period
β	Consumers' preference for the new extended warranty (FREW)
δ	Marketing size for the product
α	The expansion factor of market size caused by introducing the FREW
T	The EW coverage period set by the firm
c	The production cost of the product
c_i	The average cost of the TEW ($i=A$) or FREW ($i=B$), or the unit cost of each maintenance ($i=m$)or replacement ($i=r$)
C_E	The total fixed cost of introducing the new EW (FREW) s
θ	The cost discount factor due to the substitution of these two EWs
p	The product price
p_i	The price of the TEW ($i=A$) or FREW ($i=B$) or the bundled services ($i=AB$)
υ	The failure rate of the product at time t

N	The number of free replacement times
T_B	The actual coverage period of the FREW
T_R	The period that consumers perceive their products protected by EWs
T_N	The product's lifetime at the time of the Nth failure where N times free replacement is allowed
D_i	Total demand for the product ($i = P$), the TEW ($i = A$), the FREW ($i = B$), or the both services ($i = AB$)
Π_F	The profit of the firm

3.1 The demand functions

In our paper, the product demand is assumed to be linear. The base market size for the product is δ . The demand is determined by the product price, which is defined by $D_P = \delta - p$. When the firm introduces the new EW (i.e., FREW), considering that the variety of extended warranties can attract more consumers to purchase products and improve the sales of the product (Hartman & Laksana, 2009; Jindal, 2015; Zhang *et al.*, 2020), the market size becomes $\alpha\delta$, where $\alpha > 1$. Those consumers who purchase products can choose to buy EW or not. According to Li *et al.* (2012) and Fu *et al.* (2022), the EW period plays a significant role in the consumer utility of the EWs. The longer the EW coverage period is, the higher consumer utility is. Thus, we assume that the consumer utility of the EW is linear in the consumer expected coverage period of the EW. The consumer perceived value on the per unit EW period k obeys uniform distribution from 0 to 1. Based on the above description, the perceived value for the EW service is kT_R , where T_R represents the period that consumers perceive their products under the protection of EWs. For TEW, its perceived period T_R is equal to the EW coverage period T set by firm. Thus, its consumer perceived EW value is equal to kT . For FREW, its perceived EW period T_R is related to the number of replacements provided by the firm, the product failure rate and the EW period set by firm, which is equal to $E(T_B)$. Note that, T_B is the actual expected warranty of FREW. To explore $E(T_B)$, we set T_N as the product's lifetime at the time of the Nth failure where N times free replacement is allowed. $E(T_B)$ is equal to $E(\min\{T_N, T\})$. According to Hooti *et al.* (2020) and our analysis, we can get the probability function of the actual EW period given by:

$$f_{T_B}(t) = \begin{cases} \dfrac{(\upsilon^N t^{N-1})}{\Gamma(\mathrm{N})} e^{-\upsilon t}, & 0 \le t < T \\ \displaystyle\sum_{i=0}^{N-1} \frac{(\upsilon t)^i}{i!} e^{-\upsilon t}, & t \ge T \end{cases}$$

And the expected EW period is presented as follows:

$$\begin{aligned} E(\mathrm{T}_B) = E(\min\{\mathrm{T_N}, \mathrm{T}\}) &= (N/\upsilon)(1 - \sum_{i=0}^{N} (\upsilon T)^i \mathrm{e}^{-\upsilon T} / i!) + T \sum_{i=0}^{N-1} (\upsilon T)^i \mathrm{e}^{-\upsilon T} / i! \\ &= N/\upsilon - \upsilon^{N-1} T^N / (N-1)! + (T - N/\upsilon) \sum_{i=0}^{N-1} (\upsilon T)^i e^{-\upsilon T} / i! \end{aligned}$$

What's more, due to the attractiveness of the replacement (i.e., owning a totally new product instead of the old, damaged one), consumers prefer this new EW, thus, we use β to represent the consumer preference for the FREW, which is larger than 1. Thus, its consumer perceived EW value is equal to $kE(T_B)\beta$. For the consumer who own both EWs, the perceived period of the EW service is equal to T . Thus, its consumer perceived EW value is equal to $kT\beta$. According to the above analysis, the demand functions of three cases are presented as follows.

In case O, the firm sells the product and provides the TEW only. A consumer who already purchases products and decides to buy the TEW will obtain utility $u_A^O = kT - p_A$. If $u_A^O > 0$, the consumer will purchase this TEW. The base market size of the EW is equal to the product demand. So, the $1 - p_A / T$ of product owners will purchase EW. The demand functions of the product and TEW are presented as follows:

$$D_P^O = \delta - p \tag{1}$$

$$D_A^O = (\delta - p)(1 - p_A / T) \tag{2}$$

In the case of TS, the firm introduces the new EW. Consumers who purchase products can buy nothing, TEW, FREW, or both EWs. A consumer can obtain the utility $u_A^{TS} = kT - p_A$ from buying TEW, the utility $u_B^{TS} = kE(T_B)\beta - p_B$ from buying FREW, and the utility $u_{AB}^{TS} = kT\beta - p_B - p_A$ from buying two EWs. It is because if consumers purchase both EWs, they not only can experience the free replacement services but also their product can be protected by EW during $[0, T]$. Consumers with utilities satisfying $u_A^{TS} \geq \max\{u_B^{TS}, u_{AB}^{TS}, 0\}$ will purchase TEW only, that is, $p_A / T \leq k < (p_B - p_A)/(E(T_B)\beta - T)$. Those with $u_B^{TS} \geq \max\{u_A^{TS}, u_{AB}^{TS}, 0\}$ will purchase FREW only, that is, $(p_B - p_A)/(E(T_B)\beta - T) \leq k < p_A / (\beta(T - E(T_B)))$. Besides, those consumers with $u_{AB}^{TS} \geq \max\{u_A^{TS}, u_B^{TS}, 0\}$ will purchase both extended warranties, that is, $p_A / (\beta(T - E(T_B))) \leq k \leq 1$. Therefore, the demands for the product and both EWs are given as follows:

$$D_P^{TS} = \alpha\delta - p \tag{3}$$
$$D_A^{TS} = (-E(T_B)(T - E(T_B))(T - p_A)\beta^2 + T(T^2 - (p_B + E(T_B))T + E(T_B) \tag{4}$$
$$(p_A + p_B))\beta - T^2 p_A)(\alpha\delta - p)/((T - E(T_B)\beta)T\beta(T - E(T_B)))$$
$$D_B^{TS} = (\delta\alpha - p)(\beta E(T_B) - T + p_A - p_B)/(E(T_B)\beta - T) \tag{5}$$

In the case of TB, the firm not only introduces the new EW, but also sells this service by bundling it up with the TEW. At that time, the consumer can choose the TEW or the bundle which includes TEW and FREW. Analogously, the consumer can obtain the utility $u_A^{TB} = kT - p_A$ from purchasing TEW, and the utility $u_{AB}^{TB} = kT\beta - p_{AB}$ from purchasing the bundle. When $u_A^{TB} \geq \max\{u_{AB}^{TB}, 0\}$, that is, $p_A / T \leq k < (p_{AB} - p_A)/(T(\beta - 1))$, the consumer prefers TEW. When $u_{AB}^{TB} \geq \max\{u_A^{TB}, 0\}$, that is, $(p_{AB} - p_A)/(T(\beta - 1)) \leq k \leq 1$, the consumer prefers the bundle. Thus, the demands for the product and both EWs are given as follows:

$$D_P^{TB} = \alpha\delta - p \tag{6}$$
$$D_A^{TB} = (\delta\alpha - p)(\beta p_A - p_{AB})/(T(\beta - 1)) \tag{7}$$
$$D_{AB}^{TB} = (\delta\alpha - p)(T(\beta - 1) - p_{AB} + p_A)/(T(\beta - 1)) \tag{8}$$

Note that, based on the reality that the free replacement has limited attractiveness, and to ensure the demand for both services, we assume $\beta \in \left(T / \mathrm{E}(\mathrm{T_B}), \mathrm{T}/(\mathrm{T} - \mathrm{E}(\mathrm{T_B}))\right)$.

3.2 The cost structures

The above analysis shows us that the number of product failures obeys HPP with rate $\Lambda(\mathrm{t}) = \int_0^T \upsilon dt = \upsilon T$. Thus, the expected failure number in the EW period $[0, T]$ is υT. When the firm provides the TEW, each maintenance of the damaged product during the EW period will incur the firm's cost c_m. The average maintenance cost per unit service sold is $c_A = c_m \upsilon T$. When the firm decides to introduce the FREW, it should bear the cost C_E of introducing the new EW and the cost per replacement c_r. The unit cost of the FREW c_B is the per replacement cost multiplied by the expected value of the free replacement times $E(N)$, that is, $c_r E(N)$. The expected free replacement times $E(N)$ is derived as follows:

$$E(N) = \sum_{i=1}^{N} iP\{n(T) = \mathrm{i}\} + NP\{n(T) > N\}$$
$$= \sum_{i=0}^{N} (\upsilon T)^i e^{-\lambda T} / (i-1)! + N(1 - \sum_{i=0}^{N} (\upsilon T)^i e^{-\lambda T} / i!)$$

If the consumers own both extended warranties, they can choose to maintain or replace their damaged products, and the firm can save some cost. Due to the cost savings and the attractiveness of the free replacement, the unit cost of providing two EWs for one consumer is $\theta c_m \upsilon T + c_r E(N)$, where the cost-saving factor is defined by θ.

3.3 Profit models

In our paper, the product and the EWs are provided and sold by the firm. The firm considers what services to sell and makes pricing decisions on the product and services. Firstly, the firm should decide whether to introduce FREW for its product in view of its benefit and cost. Then, if the firm provides the FREW, how to sell the service, that is, bundle sale or sold separately, should be thought over. After that, it should make prices for both the product and services.
According to the proposed problem, we could present the profit functions in the cases of O, TS, and TB, respectively.
In the case of O, the profit function of the firm is presented as follows:

$$\Pi_F^O(p, p_A) = (p - c)D_P^O + D_A^O(p_A - c_A) \tag{9}$$

In the case of TS, the profit function of the firm is presented as follows:

$$\Pi_F^{TS}(p, p_A, p_B) = (p-c)D_P^{TS} + D_A^{TS}(p_A - c_A) + D_B^{TS}(p_B - c_B) + D_{AB}^{TS}(p_A + p_B - c_B - \theta c_A) - C_E \tag{10}$$

In the case of TB, the profit function of the firm is presented as follows:

$$\Pi_F^{TB}(p, p_A, p_{AB}) = (p-c)D_P^{TB} + D_A^{TB}(p_A - c_A) + D_{AB}^{TB}(p_{AB} - \theta c_A - c_B) - C_E \tag{11}$$

4. Equilibrium analysis

In this section, we show the optimal decisions of the three cases in Table 2. What's more, we will answer the issues raised in our paper---Should the firm provide the FREW for the consumers? If it does, how should it sell the service---bundled or sold separately? Also, we will explore the effect of some factors on the firm's strategies. First, we follow of interest the service selection strategy.

Table 2
The optimal solutions of base model

Case	**The optimal solutions**
Case O	$p^{O*} = -(c_m\upsilon - 1)^2 T/8 + c/2 + \delta/2$. $p_A^{O*} = (c_m\upsilon + 1)T/2$.
Case TS	$p^{TS*} = (E(T_B))^2(T - E(T_B))^2\beta^4 - 4((-1/2 - c_m{}^2\upsilon^2/4 + \upsilon(\theta - 1/2)c_m)T^2$ $+((1/4 + c_m{}^2\upsilon^2/4 - \upsilon(\theta - 1/2)c_m)E(T_B) + \alpha\delta + E(N)c_r/2 + c)T - (\alpha\delta$ $+E(N)c_r/2 + c)E(T_B))(T - E(T_B))E(T_B)\beta^3 + 4((-3/4 + \upsilon(-1/2 + \theta)$ $c_m)T^3 + ((1/2\theta c_m{}^2\upsilon^2 - \theta c_m\upsilon + 1/4)E(T_B) - c_m c_r\upsilon E(N)/2 + \alpha\delta + E(N)$ $c_r/2 + c)T^2 + ((1/2c_m c_r\upsilon E(N) - 2\alpha\delta - E(N)c_r - 2c)E(T_B) + E(N)^2 c_r{}^2$ $/4)T - E(N)^2 c_r{}^2 E(T_B)/4)(T - E(T_B))\beta^2 - 2((-1/2 + \upsilon^2(-1/2 + \theta)c_m{}^2$ $-c_m\upsilon)T^3 + ((1/2 - \upsilon^2(\theta^2 + 2\theta - 1)c_m{}^2/2 + c_m\upsilon)E(T_B) + c_m c_r\upsilon E(N) - 2\alpha\delta$ $-E(N)c_r - 2c)T^2 + ((-E(N)c_m c_r\upsilon + E(N)cr + 2\alpha\delta + 2c)E(T_B) - E(N)^2$ $c_r{}^2/2)T + E(N)^2 c_r{}^2 E(T_B)/2)T\beta - T^4 c_m{}^2\upsilon^2\theta^2 + T\beta)/(8\beta((T - E(T_B))\beta$ $+T)(T - E(T_B)\beta)(T - E(T_B)))$ $p_A^{TS*} = ((c_m\upsilon + 2)(T - E(\mathrm{T_B}))\beta + Tc_m\upsilon\theta)T/(2(T - E(\mathrm{T_B}))\beta + 2T)$ $p_B^{TS*} = ((\beta - 1 + (\theta - 1)\upsilon c_m)T^2 + (E(N)\beta c_r + E(T_B)\beta^2 + E(N)\ c_r)T$ $-E(N)\ E(T_B)\beta c_r - E(T_B)^2\beta^2)/((2\beta + 2)T - 2E(T_B)\beta)$
Case TB	$p^{TB*} = ((-\beta^2 + (-c_m{}^2\upsilon^2 + 2c_m\theta\upsilon + 1)\beta - \upsilon(2 + \upsilon(\theta - 2)c_m)c_m\theta)T^2 + ((2E(N)c_r + 4\alpha\delta$ $+4c)\beta - 2c_r\upsilon E(N)(\theta - 1)c_m - 4\alpha\delta - 2E(N)c_r - 4c)T - E(N)^2 c_r{}^2)/(8T(\beta - 1))$ $p_A^{TB*} = (c_m\upsilon + 1)T/2$ $p_{AB}^{TB*} = (\theta c_m\upsilon T + T\beta + c_r E(N))/2$

4.1 The optimal EW strategy

Proposition 1. The size relationships between the product price of the three models are presented as follows:

(a) When $\theta > \tilde{\theta}$ and $c_m > \tilde{c}_m$, if $\alpha < \alpha''$, we have $p^{TS*} < p^{TB*} < p^{O*}$; if $\alpha'' < \alpha < \alpha'$, we have $p^{TS*} < p^{O*} < p^{TB*}$; if $\alpha' < \alpha$, we have $p^{O*} < p^{TS*} < p^{TB*}$.
(b) otherwise, if $\alpha < \alpha'$, we have $p^{TB*} < p^{TS*} < p^{O*}$; if $\alpha' < \alpha < \alpha''$, we have $p^{TB*} < p^{O*} < p^{TS*}$; if $\alpha > \alpha''$, we have $p^{O*} < p^{TB*} < p^{TS*}$.

Proposition 1 (a) states that if the cost discount is relatively low and the unit maintenance cost of TEW is relatively high, usually when the expansion factor of market size is relatively low, the product price in case TS is the lowest and that in case O is highest. With the increase of the expansion factor, the product price in Case TB is the highest. Eventually, the product price in case O is the lowest. Owing to the high maintenance cost and the low discount, the firm in case TB is more willing to set a high product price than in case TS, which can select high-quality customers, increases the margin profit from the product and reduces the potential EW market to control the cost incurred by providing TEW. Moreover, with the increase of the expansion factor, the product in case TB or TS is more attractive, leading to a larger increase in product price in these two cases than in case O.

Proposition 1 (b) shows that when the unit maintenance cost is relatively low or the cost discount in TEW is relatively high, if the expansion factor of market size is relatively low, the product price in case TB is still the lowest. The low maintenance cost and the high discount motivate the firm to expand the base EW market size. Once this discount factor is above a threshold, the product price in case TB would be higher than in case O due to the enormous potential product market size. Besides, as the expansion factor increases, the product price in case O is the lowest for the same reason. Significantly, under any situation, if the expansion factor of the product size caused by the FREW is low, the product price in case O should be the highest, which can help the firm control the EW cost. If the benefits of introducing FREW to the product market size are great enough, the firm is encouraged to set a higher product price regardless of whether the maintenance cost is high.

Proposition 2. The size relationships between the price of EW services are given as follows:

(a) $p_A^{O*} = p_A^{TB*} > p_A^{TS*}$;
(b) $p_{AB}^{TB*} > p_A^{TS*} + p_B^{TS*}$.
Proposition 2 (a) shows that the price of TEW, which is the same in cases O and TB, is higher than in case TS. If the FREW is introduced and sold separately, the competition leads to the lower price of TEW in case TS, and the relatively high price of service A (TEW) entices consumers to purchase the bundled package in case TB. It is an interesting story presented in Proposition 2 (b) that the optimal price of the bundled product will be higher than the sum of both EWs' prices in case TS. Because, in the case of TB, those consumers whom FREW attracts can bear the higher bundle service price due to the high price of its competitors. In case TS, the competition between the TEW and the FREW lowers the EWs price. Thus, the bundle service price is higher than the sum of both EWs in case TS.

Proposition 3. The product demands under three cases meet the following rules:

(a) when $\theta > \tilde{\theta}$ and $c_m > \tilde{c}_m$, we still have $D_p^{O*} < D_p^{TB*} < D_p^{TS*}$;
(b) otherwise, if $\alpha < \tilde{\alpha}$,we have $D_p^{TS*} < D_p^{O*} < D_p^{TB*}$; if $\alpha > \tilde{\alpha}$, we have $D_p^{O*} < D_p^{TS*} < D_p^{TB*}$.
Proposition 3 (a) shows that when the cost discount is relatively low and the unit maintenance cost is relatively high, the product demand in case TS is still the highest, while that in case O is the lowest. From Proposition 1, we know that when the expansion factor is relatively small, the product price plays a vital role in the product demand. With the increase of the expansion factor, despite the product price increases, the benefit of the FREW in the potential market size can make up for these losses. Thus, the product demand in case O is the lowest.

Proposition 3 (b) shows that if the saving effect on the TEW cost of the FREW is high or the maintenance cost of the TEW is relatively low, the product demand in case TB is still the highest for its lowest product price among the three cases. As for the size relationship of product demands in cases O and TS, it is affected by the expansion factor. Like Proposition 3 (a), product demand is affected by the expansion effect on the product market size and the product prices.
Proposition 3 implies that even if the product price is high, the expansion effect of the FREW on the product market size is high enough, the product demand in case TB and TS would still be higher than in case O.

Proposition 4. The relationships between the proportion of the different EWs in a potential market of EW under three cases are given as follows:

(a) When $c_m < \hat{c}_m$, we have, $(D_A^{TS*} + D_{AB}^{TS*}) / D_P^{TS*} > D_A^{O*} / D_P^{O*} = (D_A^{TB*} + D_{AB}^{TB*}) / D_P^{TB*}$; otherwise, we have $D_A^{O*} / D_P^{O*} = (D_A^{TB*} + D_{AB}^{TB*}) / D_P^{TB*} > (D_A^{TS*} + D_{AB}^{TS*}) / D_P^{TS*}$;
(b) when $c_m < \hat{c}_m$, we have $(D_B^{TS*} + D_{AB}^{TS*}) / D_P^{TS*} < D_{AB}^{TB*} / D_P^{TB*}$; otherwise, we have $(D_B^{TS*} + D_{AB}^{TS*}) / D_P^{TS*} > D_{AB}^{TB*} / D_P^{TB*}$.

According to Proposition 4, when the unit maintenance cost is relatively low, the proportion of TEW owners in case TS is higher than that in case O and TB, caused by the low price of TEW in case TS. This means that the proportion of FREW in case TS is lower. With the maintenance cost increase, the proportion of TEW owners in case TS decreases, owing to the shift of consumer choice from TEW to FREW.

Proposition 4 implies that the cost of the TEW will affect consumer structure, but the base market of the EW (i.e., the product demand) influences the actual EW demand. Combined with Proposition 3, when the maintenance cost is relatively low, the firm in case TB is more willing to reduce the product price to promote the demand for service A (TEW), which has a high unit profit. In contrast, when the maintenance cost is high, the lower margin profit of both EWs in case TS encourages the firm to promote the base EW market. Thus, the product price is also an effective way for the firm to control the EWs market.

Theorem 1. The optimal new EW introduction strategy is presented as follows:

(a) When $\alpha > \tilde{\alpha}$, and $C_E < C_E{'}$, we have $\Pi_F^{TS*} > \Pi_F^{O*}$; otherwise, we have $\Pi_F^{TS*} < \Pi_F^{O*}$;

(b) When $C_E < C_E{''}$, we have $\Pi_F^{TB*} > \Pi_F^{O*}$; otherwise, we have $\Pi_F^{TB*} < \Pi_F^{O*}$.

Theorem 1 (a) shows that only when the expansion factor of market size caused by the FREW is larger than a threshold and the total fixed cost of providing the FREW is relatively small, the firm will introduce the FREW; otherwise, the firm prefers not to develop the new service. Due to the encroachment and the relatively high cost of the new service, the firm's profit may fall. Suppose the expansion of market size is large enough, which means the potential market for extended warranties expands. In that case, the firm can profit more from introducing the FREW, which sells separately. Besides, the development cost of the FREW should be considered. If the new service burdens the firm and depresses its profit, there is no reason to introduce it.

Theorem 1 (b) shows that only when the development cost of the FREW is lower than a threshold, will the firm provide the FREW through bundling with the TEW. When the FREW is sold as a bundle, which will reduce the total maintenance cost of TEW and increase product and EWs demand, the firm will introduce the FREW that bundle with the TEW if the development cost can be borne.

Theorem 1 implies that the firm selling new EW separately should care about the expansion effect brought by this service, although the development cost is low. But if the firm adopts a bundle strategy, it only needs to trade off the development costs and profits.

Theorem 2. The optimal sale method of the FREW is presented as follows:
(a) When $\theta < \tilde{\theta}$, there still exists $\Pi_F^{TS*} < \Pi_F^{TB*}$;
(b) otherwise, if $c_m < \tilde{c}_m$, there exists $\Pi_F^{TS*} < \Pi_F^{TB*}$, if $c_m > \tilde{c}_m$, there exists $\Pi_F^{TS*} > \Pi_F^{TB*}$.

According to Theorem 2(a), we can know that when the cost discount of the TEW due to some consumers who own both TEW and FREW is relatively small, the firm's profit will be higher in case TB than that in case TS. It is because that the product demand (i.e., the EW base market) is larger and the prices of both EWs are higher in case TB.

As for Theorem 2 (b), it shows that except that the cost discount is considerable, usually, when the unit maintenance cost is relatively high, the firm is more willing to sell the FREW separately; when the unit maintenance cost is relatively low, the firm prefers to sell the bundled service. Because if the maintenance cost is high, a higher percentage of product owners will buy FREWs and a lower percentage will buy TEWs that can bring limited profit in case TS, as shown in Proposition 4. Also, the base market size of EW is larger at the same time. Thus, the firm's profit in this case is higher. If the maintenance cost is relatively small, the higher unit EW profit encourages the firm to adopt the bundle strategy.

Theorem 2 implies that when the firm selects the sale method of the FREW, it should consider the maintenance cost if the new services cannot bring good benefits.

Lemma 1. The optimal EW strategy of the firm can be concluded as follows:

(a) When $\theta < \tilde{\theta}$, if $C_E < C_E{''}$, TB outperforms both TS and O; otherwise, O outperforms both TB and TS.

(b) when $\theta > \tilde{\theta}$ and $c_m < \tilde{c}_m$, if $C_E < C_E{''}$, TB outperforms both TS and O; otherwise, O outperforms both TB and TS.

(c) when $\theta > \tilde{\theta}$ and $c_m > \tilde{c}_m$, if $C_E < C_E{'}$, TS outperforms both TB and O; otherwise, O outperforms both TB and TS.

Lemma 1 states that when the cost discount caused by the substitutability of these two EWs creates a suitable environment for the bundled EW, we only should care about the development cost of this service. Otherwise, the optimal EW strategy is related to the unit maintenance cost of TEW and the development cost of the FREW. In any case, if the development cost is too high, the firm has better maintain the status quo.

To present this Lemma clearly, we do some numerical examples. Firstly, we set $T = 0.6$, $N = 1$, $\nu = 0.15$, $\beta = 2$, and $\alpha = 1.2$ for the first and second examples. But in the first one, we set $\delta = 1$, $c = 0.8$, $\theta = 0.1$, $c_r = 0.7$ and increase c_m from 0 to 0.8. In the second one, $\delta = 3$, $c = 2.5$, $\theta = 0.95$, $c_r = 2$ are set and c_m increases from 0 to 2.5. The results are shown in Figure 1 and Figure 2, respectively.

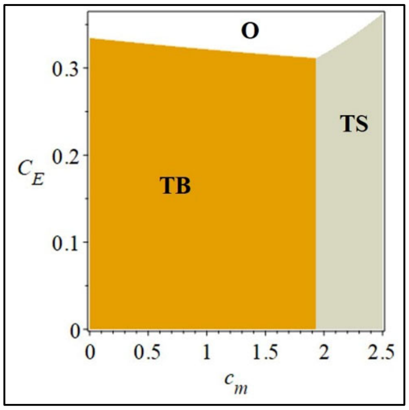

Fig. 1. The effect of c_m and C_E ($\theta < \tilde{\theta}$)

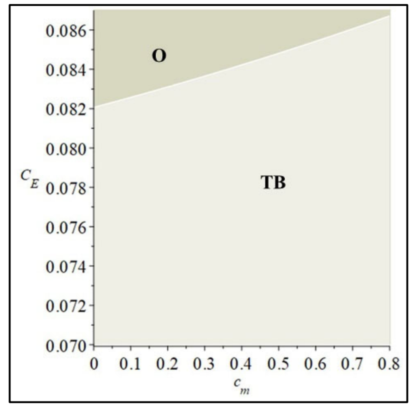

Fig. 2. The effect of c_m and C_E ($\theta > \tilde{\theta}$)

Fig. 1 shows a special situation that the cost discount is high, at that time the firm develops a strategy based on the fixed cost C_E and always bundles FREW. Fig. 2 shows that except C_E, c_m needs to be considered when making EW strategy if the cost discount is low. These figures verify our conclusion in Lemma 1.

Proposition 5. The effect of the unit replacement costs on the sales method selection is given as follows: $\partial \tilde{c}_m / \partial c_r > 0$

Proposition 5 shows that as the unit replacement costs increase, the firm is more likely to sell the EW package. Because if the unit replacement cost is relatively high, the profit from selling the FREW separately is limited, the firm prefers to sell the bundled product, which can bring more profit due to the higher bundled product price shown in Proposition 2.

Beyond question, the number of free replacement times and the product failure rate play a major role in the firm's EW strategies. We will explore the impact of these factors in the next section.

4.2 The influences of the number of free replacement times and the product failure rate

In this section, we will do some numerical examples to explore how the number of free replacement times and product failure rate affect the firm's EW strategies.

Proposition 6. The effect of T, υ, and N on the expected EW coverage period and the average replacement times during the EW period are given as follows:

(a) $\partial E(\mathrm{T_B}) / \partial \upsilon < 0$, $\partial E(\mathrm{T_B}) / \partial N > 0$, $\partial E(\mathrm{T_B}) / \partial T > 0$;
(b) $\partial E(N) / \partial \upsilon > 0$, $\partial E(N) / \partial N > 0$, $\partial E(N) / \partial T > 0$.

According to Proposition 6, we can know that the expected EW coverage period and the average replacement times increase with the number of free replacement times or the EW coverage period. But as the failure rate increases, the expected EW period decreases, and the average replacement times increases. Thus, the high quality of the product positively impacts cost-saving and consumer-perceived value.

Proposition 7. The effect of υ and N on the optimal decisions are presented as follows:

(a) $\partial p_A^{O*} / \partial \upsilon > 0$, $\partial p_A^{TS*} / \partial \upsilon > 0$, $\partial p_A^{TB*} / \partial \upsilon > 0$, $\partial p_{AB}^{TB*} / \partial \upsilon > 0$;
(b) $\partial p_A^{TS*} / \partial N < 0$, $\partial p_{AB}^{TB*} / \partial N > 0$.

Proposition 7 shows that the optimal prices of service A(TEW) in all three cases still increase with the failure rate for the high maintenance cost. Besides, the optimal bundled EW price increases with the failure rate for the same reason. Moreover, the increase of N has a positive effect on the optimal bundled EW price, a negative effect on the optimal TEW price in case TS, and no effect on the optimal TEW price in case TB. The firm would pay more replacement costs if N increases. So in case TS, the firm will set a relatively low price for the TEW so as to encourage customers to purchase it. In case TB, the consumer who wants to purchase FREW should fully bear its high cost. However, what are these two factors' effects on the optimal price of the FREW in case TS?

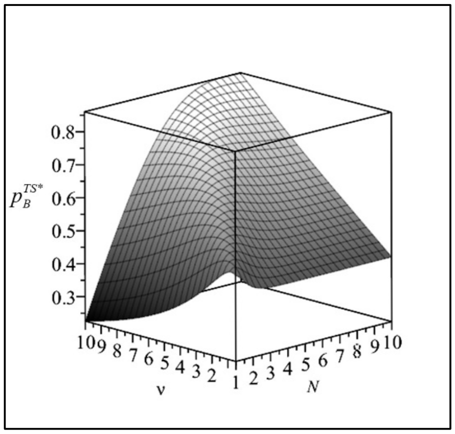

Fig. 3. The effects of υ and N on p_B^{TS*} (base)

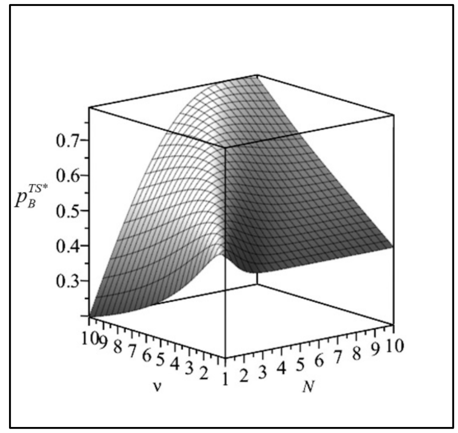

Fig. 4. The effects of υ and N on p_B^{TS*} ($c_m = 3/20$)

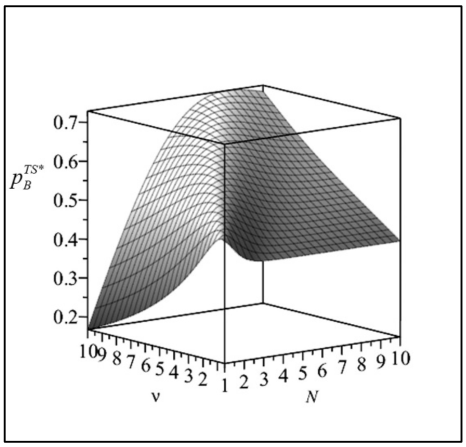

Fig. 5. The effects of υ and N on p_B^{TS*} ($c_m = 1/5$)

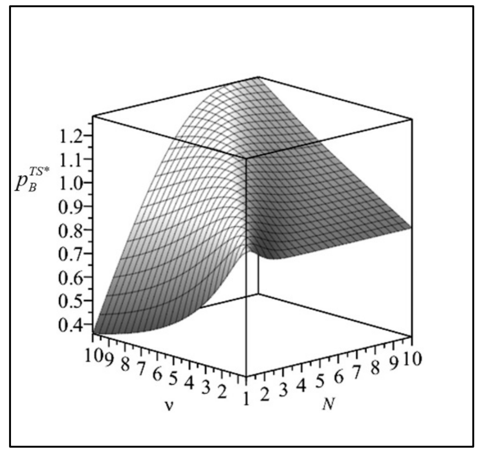

Fig. 6. The effects of υ and N on p_B^{TS*} ($\beta = 3$)

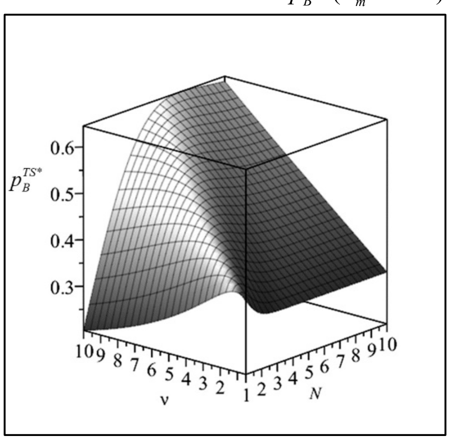

Fig. 7. The effects of υ and N on p_B^{TS*} ($T = 0.6$)

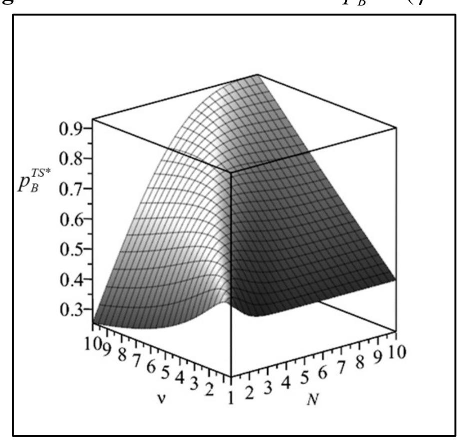

Fig. 8. The effects of υ and N on p_B^{TS*} ($\theta = 0.8$)

To investigate it, we do some numerical examples. Firstly, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $c_m = 0.1$, and $c_r = 0.15$. In the second example, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $c_m = 0.15$, and $c_r = 0.15$. We set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $c_m = 0.2$, and $c_r = 0.15$ in the third examples. In the fourth one, we set $T = 0.8$, $\theta = 0.6$, $\beta = 3$, $c_m = 0.1$, and $c_r = 0.15$. Next, we set $T = 0.6$, $\theta = 0.6$, $\beta = 2$, $c_m = 0.1$, and $c_r = 0.15$. Finally, $T = 0.8$, $\theta = 0.8$, $\beta = 2$, $c_m = 0.1$, and $c_r = 0.15$ Based on the reality, we vary υ from 0.5 to 10, and vary N from 1 to 10 for all above examples, the results of the effects of υ and N on p_B^{TS*} are shown in Figs. 3-8. From the above numerical examples, we know that when N is close to υT, the price of the FREW will be the highest. Whether the number of free replacement times is too high or too low, the price of the FREW will be low. It is because the FREW price is mainly related to its cost and its consumer-perceived value. The relatively high N can attract more consumers to purchase the FREW. Once the FREW cost can be covered, the firm would prefer to promote this service demand. From Figs. 3-8, the highest p_B^{TS*} appears when N is close to υT, which proves the above inferences. Therefore, the firm is advised to set an appropriate N to increase profit. To observe the effects of υ and N on the sales methods of the EWs, we also present some numerical examples. Firstly, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $N = 1$, and $c_r = 0.15$ in Fig. 9 (a). To ensure the existence of the optimal solutions, we vary υ from 0.1 to 2. Then, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $N = 2$, $c_r = 0.15$ in Fig. 9 (b) and vary υ from 0.1 to 5. In next examples, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $N = 3$, $c_r = 0.15$ and vary υ from 0.1 to 8. The result is shown in Fig. 9 (c). At last, in Fig. 9 (d), we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $N = 4$, $c_r = 0.15$ and vary υ from 0.1 to 10. All results are shown in Fig. 9. Then, we explore the impact of the N on the $\tilde{c}_m$. In Fig. 10 (a), we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $\upsilon = 1$, and $c_r = 0.15$. Then, we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $\upsilon = 5$, and $c_r = 0.15$ in Fig. 10 (b). In the last one is presented in Fig. 10 (c), we set $T = 0.8$, $\theta = 0.6$, $\beta = 2$, $\upsilon = 10$, and $c_r = 0.15$. In all above numerical examples, we vary N from 1 to 10. Fig. 10 is presented as follows.

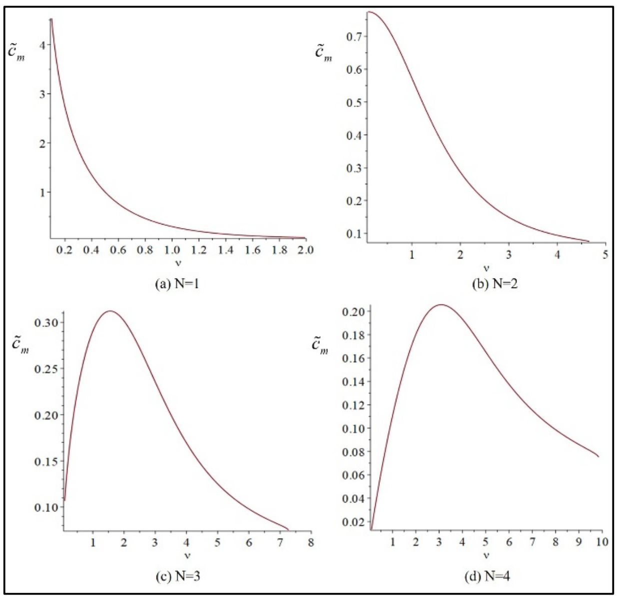

Fig. 9. The effect of υ on the $\tilde{c}_m$

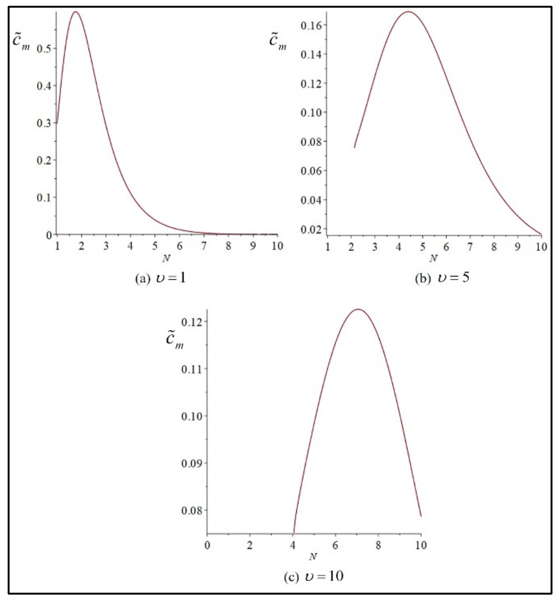

Fig. 10. The effect of N on the $\tilde{c}_m$

The above figures illustrate that when N is low, the firm is more likely to sell EWs separately as υ increases. But if N is high, with the increase of υ, $\tilde{c}_m$ increases first, and then goes down. Because when υ is lower than a threshold, which is close to N/T the firm can save EW cost by bundling strategy. Once it exceeds this threshold, selling EWs separately can avoid high maintenance cost from TEW. Focusing on the effect of N on $\tilde{c}_m$, the firm's willingness to the bundling sale rises at first and then drops as the failure rate grows. The more replacement times offered by the firm will contribute to the consumer demand for the FREW, while bringing more costs to the firm. If the FREW can bring considerable profits, the firm is more willing to sell it separately.

In general, the number of free replacement times and the failure rate of the product should be deliberated, which are critical factors in sales method selection. What's more, the optimal N will be different in different cases.

5. Extension

5.1 Considering the two-tier supply chain

In reality, there exists a lot of e-retailers who purchase the products from the manufacturers and provide the EW by themselves, such as JD, Amazon, and Suning (Amazon.com, 2022; JD.com, 2022; Suning.com, 2022). They play important roles in the e-commerce context (Ozbilge *et al.*, 2022; Shi *et al.*, 2023) and always provided a variety of EWs, including FREW, TEW, and other EWs. Besides, it's common for them to sell package EWs. Thus, we consider the two-tier supply chain consisting of a manufacturer and a retailer in this section. The retailer purchases products from the manufacturer and sells the product and the EWs. Except for the profit function, other settings are like the basic model. Note that, the wholesale price in this paper is defined as w, then we use SO, STS, and STB to represent the O, TS, and TB of the basic model, respectively.
In case SO, the manufacturer and the retailer's profit functions are presented as follows:

$$\Pi_M^{SO} = (w - c) D_p^{SO} \tag{12}$$

$$\Pi_R^{SO} = (p-c)D_P^{SO} + D_A^{SO}(p_A - c_A) \tag{13}$$

In case STS, the manufacturer and the retailer's profit functions are presented as follows:

$$\Pi_M^{STS} = (w-c)D_p^{STS} \tag{14}$$

$$\Pi_R^{STS} = (p-w)D_P^{STS} + D_A^{STS}(p_A - c_A) + D_B^{STS}(p_B - c_B) + D_{AB}^{STS}(p_A + p_B - c_B - \theta c_A) - C_E \tag{15}$$

In case STB, the manufacturer and the retailer's profit functions are presented as follows:

$$\Pi_M^{STB} = (w-c)D_p^{STB} \tag{16}$$

$$\Pi_R^{STB} = (p-w)D_P^{STB} + D_A^{STB}(p_A - c_A) + D_{AB}^{STB}(p_{AB} - \theta c_A - c_B) - C_E \tag{17}$$

The optimal solutions are given in Appendix.

Theorem 3. The optimal EW strategy of the retailer in the supply chain case is presented as follows:

(a) When $\alpha > \tilde{\alpha}$, and $C_E < C_E' / 4$, we have $\Pi_R^{STS*} > \Pi_R^{SO*}$; otherwise, we have $\Pi_R^{STS*} < \Pi_R^{SO*}$;

(b) When $C_E < C_E'' / 4$, we have $\Pi_R^{STB*} > \Pi_R^{SO*}$; otherwise, we have $\Pi_R^{STB*} < \Pi_R^{SO*}$.

Like Theorem 1, only when the expansion factor is relatively high, and the development cost is relatively low, the retailer will introduce FREW, sell it separately. What's more, if the retailer chooses to sell the EW package, its decision is affected solely by the development cost. If the development cost is relatively low, the retailer prefers to sell the bundled EW. Theorem 3 implies that the optimal EW development strategies are still held considering the two-tier supply chain.

Theorem 4. The optimal sales method of the FREW in the supply chain case is presented as follows:

(a) When $\theta < \tilde{\theta}$, there still exists $\Pi_R^{STS*} < \Pi_R^{STB*}$;

(b) otherwise, if $c_m < \tilde{c}_m$, there exists $\Pi_R^{STS*} < \Pi_R^{STB*}$, if $c_m > \tilde{c}_m$, there exists $\Pi_R^{STS*} > \Pi_R^{STB*}$.

Theorem 4 shows that the optimal EW sale methods are related to the unit cost of the TEW. But if the firm can gain more benefit from the consumers who own the EW package, it will still prefer to sell the bundled EW rather than sell the FREW separately. With the two-tier supply chain to consider, the optimal EW sale methods are still hold.

Lemma 2. The optimal EW strategy of the retailer in the supply chain case can be concluded as follows:

(a) When $\theta < \tilde{\theta}$, if $C_E < C_E'' / 4$, STB outperforms both STS and SO; otherwise, SO outperforms both STB and STS.

(b) when $\theta > \tilde{\theta}$ and $c_m < \tilde{c}_m$, if $C_E < C_E'' / 4$, STB outperforms both STS and SO; otherwise, SO outperforms both STB and STS.

(c) when $\theta > \tilde{\theta}$ and $c_m > \tilde{c}_m$, if $C_E < C_E' / 4$, STS outperforms both STB and SO; otherwise, SO outperforms both STB and STS.

Lemma 2 holds similar results to Lemma 1. To avoid repetition, we do not say too much.

5.2 Considering the corporate social responsibility (CSR)

In this section, we want to explore the firm's EWs strategy considering that the firm takes corporate social responsibility (CSR) into consideration. According to Iyer and Soberman (2016), we set that the utility of the firm U_F is equal to $\Pi_F + \eta CS$, and the profit functions are similar to those in the basic model. Note that, the coefficient of the CSR is denoted by η . Moreover, we assume that $\eta \in (0,1)$, because the firm is not the perfect social welfare maximizer. What's more, we use RO, RTS, and RTB to represent the O, TS, and TB of the basic model, respectively.

In case RO, the consumer surplus and the utility of the firm can be presented as follows:

$$\begin{aligned} CS^{RO} &= CS_P^{RO} + CS_{EW}^{RO} = \int_{\delta - D_p^{RO}}^{\delta} D_P^{RO} dp + D_P^{RO} \int_{p_A/T}^{1} u_A^{RO} dk \\ &= (\delta - p)^2 / 2 + ((\delta - p)(T - p_A)^2) / (2T) \end{aligned} \tag{18}$$

$$\begin{aligned} U_F^{RO} &= \Pi_F^{RO} + \eta CS^{RO} \\ &= (p-c)D_P^{RO} + D_A^{RO}(p_A - c_A) + \eta\left(\int_{\delta - D_p^{RO}}^{\delta} D_P^{RO} dp + D_P^{RO} \int_{p_A/T}^{1} u_A^{RO} dk\right) \end{aligned} \tag{19}$$

In case RTS, the consumer surplus and the utility of the firm can be presented as follows:

$$CS_P^{RTS} = \int_{\delta\alpha - D_p^{RTS}}^{\delta\alpha} D_P^{RTS} dp = (\alpha\delta - p)^2 / 2 \tag{20}$$

$$\begin{aligned} CS_{EW}^{RTS} &= D_P^{RTS}\left(\int_{p_A/\beta(T-E(T_B))}^{1} u_{AB}^{RTS} dk + \int_{(p_B-p_A)/(\beta E(T_B)-T)}^{p_A/\beta(T-E(T_B))} u_B^{RTS} dk + \int_{p_A/T}^{(p_B-p_A)/(\beta E(T_B)-T)} u_A^{RTS} dk\right) \\ &= (\alpha\delta - p)(((-E(T_B)\beta p_A + Tp_B)^2)/(2T(\beta E(T_B)-T)^2) + (((p_A \\ &-p_B)\beta - p_A)T + E(T_B)\beta p_B)(T^2 p_B - (((p_A+p_B)\beta + p_A + 2p_B)E(T_B)T)/2 + \\ &E(T_B)^2\beta(p_A + p_B/2))/(\beta(T-E(T_B))^2(T-\beta E(T_B))^2) + ((T(T-E(T_B))\beta + \\ &(-p_A - 2p_B)T + 2E(T_B)(p_A+p_B))(\beta(T-E(T_B)) - p_A))/(2\beta(T-E(T_B))^2)) \end{aligned} \tag{21}$$

$$\begin{aligned} U_F^{RTS} &= \Pi_F^{RTS} + \eta CS^{RTS} \\ &= (p-c)D_P^{RTS} + D_A^{RTS}(p_A - c_A) + D_B^{RTS}(p_B - c_B) + D_{AB}^{RTS}(p_A + p_B - c_B - \theta c_A) - C_E + \eta(CS_P^{RTS} + CS_{EW}^{RTS}) \end{aligned} \tag{22}$$

In case RTB, the consumer surplus and the utility of the firm can be presented as follows:

$$CS_P^{RTB} = \int_{\delta\alpha - D_p^{RTB}}^{\delta\alpha} D_P^{RTB} dp = (\alpha\delta - p)^2 / 2 \tag{23}$$

$$\begin{aligned} CS_{EW}^{RTB} &= D_P^{RTB}\left(\int_{(p_B-p_A)/(T(\beta-1))}^{1} u_{AB}^{RTB} dk + \int_{p_A/T}^{(p_B-p_A)/(T(\beta-1))} u_A^{RTB} dk\right) \\ &= ((T^2\beta^2 + (-T^2 - 2Tp_{AB} + p_A{}^2)\beta + 2p_{AB}(T - p_A + p_{AB}/2))(\alpha\delta - p))/(2T(\beta-1)) \end{aligned} \tag{24}$$

$$\begin{aligned} U_F^{RTB} &= \Pi_F^{RTB} + \eta CS^{RTB} \\ &= ((p-c)D_P^{RTB} + D_A^{RTB}(p_A - c_A) + D_{AB}^{RTB}(p_{AB} - \theta c_A - c_B) - C_E + \eta(CS_P^{RTB} + CS_{EW}^{RTB}) \end{aligned} \tag{25}$$

The optimal solutions are given in Appendix.

Theorem 5. The optimal EW strategy of the firm in the CSR case is presented as follows:

(a) When $\alpha > \tilde{\alpha}^R$, and $C_E < C_E^{R\prime}$, we have $U_F^{RTS*} > U_F^{RO*}$; otherwise, we have $U_F^{RTS*} < U_F^{RO*}$;

(b) When $C_E < C_E^{R\prime\prime}$, we have $\Pi_F^{RTB*} > \Pi_F^{RO*}$; otherwise, we have $U_F^{RTB*} < U_F^{RO*}$.

Theorem 5 shows similar conclusions to the basic model, which means that when the CSR is taken into consideration, the optimal EW introduction strategy still holds.

Theorem 6. The optimal sale method of the FREW in the CSR case is presented as follows:

(a) When $\theta < \tilde{\theta}$, there still exists $U_F^{RTS*} < U_F^{RTB*}$;

(b) otherwise, if $c_m < \tilde{c}_m^R$, there exists $U_F^{RTS*} < U_F^{RTB*}$, if $c_m > \tilde{c}_m^R$, there exists $U_F^{RTS*} > U_F^{RTB*}$.

(c) $\partial\tilde{c}_m^R / \partial\eta < 0$.

From Theorem 6, we can know that when CSR is considered, the optimal EW sales method derived from the basic model is still valid. What's more, with the increase of the coefficient of the CSR, the firm is more likely to sell the FREW separately. According to Proposition 2, we can know normally the price of the bundled EWs in case TB is larger than the sum of the prices of the TEW and FREW in case TS. Hence, the more social responsibility the firm takes, the more benefit of consumers should be considered which lead to the above result.

Lemma 3. The optimal EW strategy of the firm in the CSR case can be concluded as follows:

(a) When $\theta < \tilde{\theta}$, if $C_E < C_E^{R\prime\prime}$, RTB outperforms both RTS and RO; otherwise, RO outperforms both RTB and RTS.

(b) when $\theta > \tilde{\theta}$ and $c_m < \tilde{c}_m^R$, if $C_E < C_E^{R\prime\prime}$, RTB outperforms both RTS and RO; otherwise, RO outperforms both RTB and RTS.

(c) when $\theta > \tilde{\theta}$ and $c_m > \tilde{c}_m^R$, if $C_E < C_E^{R\prime}$, RTS outperforms both RTB and RO; otherwise, RO outperforms both RTB and RTS.

Lemma 3 holds similar results to Lemma 1. To avoid repetition, we do not say too much.

6. Conclusion

In the context of consumption upgrading and diversity, to cater to the consumers and obtain more profits, firms have begun to develop various EWs. As a new EW, the FREW is widespread nowadays and more attractive than the TEW, but it may burden firms. Considering the limited resources of firms and the cannibalism effect of the FREW, the firms face the challenge

of whether to introduce this new EW, that is, the FREW. Meanwhile, bundling sales, a popular sales method, significantly affects product demands and profits. However, whether to adopt this bundling method for EWs should be deliberate.
Our paper examines the introduction and bundling problems of EWs by constructing three theoretical models that consider the product failure obeys HPP and then concludes with a pairwise comparison of profits under different models. Some significant insights about whether to introduce and how to sell the FREW are presented as follows.

The optimal introduction strategy: Normally, the introduction strategy of the FREW is related to its development cost. But when the expansion factor of market size caused by the FREW is not high enough, even if the development cost is relatively low, the firm should not sell the FREW separately. Moreover, when the development cost is too high, it is suggested to provide TEW only.

The optimal sales method of the FREW: If the firm decides to introduce the FREW, the better sales method is mainly affected by the unit maintenance cost of the TEW and the cost-saving effect of the FREW. Usually, when the unit maintenance cost is relatively high, the FREW is better to be sold separately. It is noteworthy that if the cost-saving effect of the FREW are great enough, the bundling sales strategy is still the better method. Besides, the number of free replacement times and the product failure rate contribute to the possibility of adapting different sales methods.

The optimal decisions: If the incentive function in the product market is relatively large, the product price after introducing FREW will be higher. The size relationship of the product price under different EW sales methods depends on the unit maintenance cost and the cost discount factor caused by the FREW. For the prices of different EW services, the price of the TEW in case TS is lower than those in other cases. What's more, the firm can set a higher price for the bundled EW than the sum of the TEW and the FREW's price in case TS.

Given the significance of new EWs and the popularity of bundling sales, the bundling problem in the EW area deserves further research attention. There are several directions for further exploration. The EW may attract consumers who are risk aversion. It's worthwhile to consider consumer risk aversion in the consumer utility. Moreover, consumers will return their unsatisfied products, thus it will be interesting to explore the introduction and bundling strategies of new EWs with considering consumer returns (Cao and Choi, 2022). Besides, we assume that there is only one product in the market. The results may be different if there exists competition in the product market. Finally, our paper studies the introduction and sales methods strategy of the free replacement EW. There are other new EWs in the market, such as accident protection insurance and trade-in warranty. It would be promising to explore the bundling problem of other EWs.

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Appendix A

Table A.1.
Symbol comparison table

Symbol	Expression
A	$\alpha\delta - c$
A'	$\delta - c$
B	$E(T_B) - T$
F	$E(T_B)\beta - T$
G	$E(T_B)\beta - T\beta - T$
J	$E(T_B)\beta - T\beta + T$
H	$(B(T+(\theta-1)E(T_B))\beta^2 + ((\theta+1)T^2 + (-c_r\theta E(N) + E(N)c_r - E(T_B))T$ $+E(T_B)E(N)c_r(\theta-1))\beta - T^2\theta)^2$
H^R	$(-\beta E(N)(\theta-1)\mathrm{B}c_r + J(\theta F - \beta B)(\eta-1))^2$

Table A.2
The optimal solutions of extension model considering two-tier supply chain

Case	The optimal solutions
Case SO	$w^{SO*} = (c_m \upsilon - 1)^2 T / 8 + c / 2 + \delta / 2$
	$p^{SO*} = -(c_m \upsilon - 1)^2 T / 16 + c / 4 + 3\delta / 4$
	$p_A^{SO*} = (c_m \upsilon + 1) T / 2$
Case STS	$w^{STS*} = ((\alpha\delta + c)/2) - ((E(T_B)^2 (T - E(T_B))^2 \beta^4 - 4((-1/2 - c_m{}^2 \upsilon^2 / 4 + \upsilon(\theta - 1/2) c_m) T^2 + ((1/4 + c_m{}^2 \upsilon^2 / 4 - \upsilon(\theta - 1/2) c_m) E(T_B) + E(N) c_r / 2) T - E(N) c_r / 2 E(T_B))(T - E(T_B)) E(T_B) \beta^3 + 4((-3/4 + \upsilon(-1/2 + \theta) c_m) T^3 + ((1/2 \theta c_m{}^2 \upsilon^2 - \theta c_m \upsilon + 1/4) E(T_B) - c_m c_r \upsilon E(N)/2 + E(N) c_r / 2) T^2 + ((1/2 c_m c_r \upsilon E(N) - E(N) c_r) E(T_B) + E(N)^2 c_r{}^2 / 4) T - E(N)^2 c_r{}^2 E(T_B) / 4)(T - E(T_B)) \beta^2 - 2((\upsilon^2 - 1/2 + (-1/2 + \theta) c_m{}^2 - c_m \upsilon) T^3 + ((1/2 - \upsilon^2 (\theta^2 + 2\theta - 1) c_m{}^2 / 2 + c_m \upsilon) E(T_B) + c_m c_r \upsilon E(N) - E(N) c_r) T^2 + ((-E(N) c_m c_r \upsilon + E(N) c_r) E(T_B) - E(N)^2 c_r{}^2 / 2) T + E(N)^2 c_r{}^2 E(T_B)/2) T \beta - T^4 c_m{}^2 \upsilon^2 \theta^2 + T\beta) / (8\beta((T - E(T_B))\beta + T)(T - E(T_B) \beta)(T - E(T_B))))$
	$p^{STS*} = (3\alpha\delta/4 + c/4) + ((E(T_B)^2 (T - E(T_B))^2 \beta^4 - 4((-1/2 - c_m{}^2 \upsilon^2 / 4 + \upsilon(\theta - 1/2) c_m) T^2 + ((1/4 + c_m{}^2 \upsilon^2 / 4 - \upsilon(\theta - 1/2) c_m) E(T_B) + E(N) c_r / 2) T - E(N) c_r / 2 E(T_B))(T - E(T_B)) E(T_B) \beta^3 + 4((-3/4 + \upsilon(-1/2 + \theta) c_m) T^3 + ((1/2 \theta c_m{}^2 \upsilon^2 - \theta c_m \upsilon + 1/4) E(T_B) - c_m c_r \upsilon E(N)/2 + E(N) c_r / 2) T^2 + ((1/2 c_m c_r \upsilon E(N) - E(N) c_r) E(T_B) + E(N)^2 c_r{}^2 / 4) T - E(N)^2 c_r{}^2 E(T_B) / 4)(T - E(T_B)) \beta^2 - 2((\upsilon^2 - 1/2 + (-1/2 + \theta) c_m{}^2 - c_m \upsilon) T^3 + ((1/2 - \upsilon^2 (\theta^2 + 2\theta - 1) c_m{}^2 / 2 + c_m \upsilon) E(T_B) + c_m c_r \upsilon E(N) - E(N) c_r) T^2 + ((-E(N) c_m c_r \upsilon + E(N) c_r) E(T_B) - E(N)^2 c_r{}^2 / 2) T + E(N)^2 c_r{}^2 E(T_B)/2) T \beta - T^4 c_m{}^2 \upsilon^2 \theta^2 + T\beta) / (16\beta(\beta(T - E(T_B)) + T)(T - E(T_B) \beta)(T - E(T_B))))$
	$p_A^{STS*} = ((c_m \upsilon + 2)(T - E(\mathrm{T_B}))\beta + T c_m \upsilon \theta) T / (2(T - E(\mathrm{T_B}))\beta + 2T)$
	$p_B^{STS*} = ((\beta - 1 + (\theta - 1)\upsilon c_m) T^2 + (E(N)\beta c_r + E(T_B)\beta^2 + E(N)\, c_r) T - E(N)\, E(T_B) \beta c_r - E(T_B)^2 \beta^2) / ((2\beta + 2) T - 2E(T_B)\beta)$
Case STB	$w^{STB*} = -((-\beta^2 + (-c_m{}^2 \upsilon^2 + 2 c_m \theta \upsilon + 1)\beta - \upsilon(2 + \upsilon(\theta - 2) c_m) c_m \theta) T^2 + (2E(N) c_r \beta - 2 c_r \upsilon E(N)(\theta - 1) c_m - 2E(N) c_r) T - E(N)^2 c_r{}^2) / (8T(\beta - 1)) + (\alpha\delta + c)/2$
	$p^{STB*} = (3\alpha\delta/4 + c/4) + ((-\beta^2 + (-c_m{}^2 \upsilon^2 + 2 c_m \theta \upsilon + 1)\beta - \upsilon(2 + \upsilon(\theta - 2) c_m) c_m \theta) T^2 + (2E(N) c_r \beta - 2 c_r \upsilon E(N)(\theta - 1) c_m - 2E(N) c_r) T - E(N)^2 c_r{}^2) / (16T(\beta - 1))$
	$p_A^{STB*} = (c_m \upsilon + 1) T / 2$
	$p_{AB}^{STB*} = (\theta c_m \upsilon T + T\beta + c_r E(N)) / 2$

Table A.3
The optimal solutions of extension model considering corporate social responsibility

Case	The optimal solutions
Case RO	. $p^{RO*} = (\delta(1-\eta)+c)/(2-\eta)-(((c_m\upsilon-1)^2T)/(2(2-\eta)^2))$
	$p_A^{RO*} = (c_m\upsilon+1-\eta)T/(2-\eta)$
Case RTS	$p^{RTS*} = ((\alpha\delta(\eta-1)-c)/(\eta-2))+((-E(T_B)((\eta^2-2\eta)T-E(T_B)(\eta-1)^2)(T-E(T_B))^2\beta^4$
	$+((\eta^2-2\eta)T^3+2(c_m\upsilon(\theta-1)\eta+1+c_m{}^2\nu^2/2+(-2\theta+1)c_m\upsilon)E(T_B)T^2-2((\eta^2/2+(-1+$
	$c_m\upsilon(\theta-1))\eta+1/2+(c_m\upsilon)^2/2+(1-2\theta)c_m\upsilon)E(T_B)+c_rE(N))E(T_B)T+2E(T_B)^2c_rE(N))$
	$(T-E(T_B))\beta^3-2(T-E(T_B))((\eta^2+(-2+c_m\upsilon(\theta-1))\eta+3/2+(1-2\theta)c_m\upsilon)T^3+((-1/2\eta^2$
	$+\eta-1/2-\theta c_m{}^2\upsilon^2+2\theta c_m\upsilon)E(T_B)+c_mc_r\upsilon E(N)-c_rE(N))T^2+((-E(N)c_mc_r\upsilon+2c_rE(N))$
	$E(T_B)-E(N)^2c_r{}^2/2)T+E(N)^2c_r{}^2E(T_B)/2)\beta^2-2T((-\eta^2/2+(1-c_m\upsilon(\theta-1))\eta-1/2+$
	$(\theta-1/2)c_m{}^2\upsilon^2-c_m\upsilon)T^3+((\eta^2/2+(-1+c_m\upsilon(\theta-1))\eta+1/2-(\theta^2+2\theta-1)c_m{}^2\upsilon^2/2+c_m\upsilon)$
	$E(T_B)+E(N)c_mc_r\upsilon-c_rE(N))T^2+((-E(N)c_mc_r\upsilon+c_rE(N))E(T_B)-E(N)^2c_r{}^2/2)T+E(N)^2$
	$c_r{}^2E(T_B)/2)\beta-T^4\theta^2c_m{}^2\upsilon^2)/(2\beta(T-E(T_B)\beta)((T-E(T_B))\beta+T)(\eta-2)^2(T-E(T_B))))$
	$p_A^{RTS*} = ((c_m\upsilon+2-2\eta)(T-E(\mathrm{T_B}))\beta+Tc_m\upsilon\theta)T/((2-\eta)((T-E(\mathrm{T_B}))\beta+T))$
	$p_B^{RTS*} = (((\beta-1)(\eta-1)+(1-\theta)\upsilon c_m)T^2+(-E(N)\beta c_r+E(T_B)(\eta-1)\beta^2-E(N)\ c_r)T$
	$+E(N)\ E(T_B)\beta c_r-E(T_B)^2\beta^2(\eta-1))/((\eta-2)((\beta+1)T-E(T_B)\beta))$
Case RTB	$p^{RTB*} = ((\alpha\delta(\eta-1)-c)/(\eta-2))+((-\beta^2+(-c_m{}^2\upsilon^2+2c_m\theta\upsilon+1)\beta-\upsilon(2+\upsilon(\theta-2)c_m)c_m\theta)T^2$
	$+(2E(N)c_r\beta-2c_r\upsilon E(N)(\theta-1)c_m-2E(N)c_r)T-E(N)^2c_r{}^2)/(2T(\beta-1)(\eta-2)^2)$
	$p_A^{RTB*} = (c_m\upsilon+1-\eta)T/(2-\eta)$
	$p_{AB}^{RTB*} = (\theta c_m\upsilon T-T\beta\eta+c_rE(N)+T\beta)/(2-\eta)$

Optimal Decisions

In case O, $\Pi_F^O = (p-c)D_P^O + D_A^O(p_A-c_A) = (\delta-p)((p-c)+(1-p_A/T)(p_A-c_A))$ is the firm's profit function. According to $\partial\Pi_F^O/\partial p = 0$, $\partial\Pi_F^O/\partial p_A = 0$, we can get the optimal solutions $p^{O*} = -(c_m\upsilon-1)^2T/8+c/2+\delta/2$ and $p_A^{O*} = (c_m\upsilon+1)T/2$, and then we substitute the optimal solutions into the profit function and get its Hessian matrix $\begin{bmatrix} \partial\Pi_F^{O*2}/\partial p^2 & \partial\Pi_F^{O*2}/\partial p\partial p_A \\ \partial\Pi_F^{O*2}/\partial p\partial p_A & \partial\Pi_F^{O*2}/\partial {p_A}^2 \end{bmatrix}$, which is equal to $((c_m\upsilon-1)^2T-4c+4\delta)/(2T)$. According to $p^{O*} > c$ and $p^{O*} < \alpha\delta$, we can know the above Hessian matrix is bigger than 0. We substitute the optimal solutions into the profit function, the firm's optimal profit in case O is $\Pi_F^{O*} = ((c_m\upsilon-1)^2T-4c+4\delta)^2/64$.

The solving process of case TS and Case TB are similar to that of case O. The optimal profits in case TS and case TB are presented as follows:

$$\Pi_F^{TS*} = ((-4\beta BFGA - T^3c_m{}^2\upsilon^2F\theta^2 + 2\beta Tc_m\upsilon BF(Tc_m\upsilon+2\beta B)\theta - \beta\upsilon^2TB(E(T_B)^2\beta^2 - E(T_B)T\beta^2 - T^2)c_m{}^2$$
$$+2\beta\upsilon TB(E(N)Gc_r - FJ)c_m - \beta B(E(N)^2Gc_r{}^2 - 2E(N)FGc_r + F(E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B)T\beta -$$
$$3T^2\beta + T^2)))^2/(64\beta^2B^2F^2G^2)) - C_E$$
$$\Pi_F^{TB*} = (4T(\beta-1)A + T^2c_m{}^2\theta^2\upsilon^2 - 2c_m((c_m\upsilon+\beta-1)T - E(N)c_r)\upsilon T\theta + \beta(c_m{}^2\upsilon^2+\beta-1)T^2$$
$$-2E(N)c_r(c_m\upsilon+\beta-1)T + E(N)^2c_r{}^2)^2/(64(\beta-1)^2T^2) - C_E \quad .$$

Proof of Proposition 1

We compare the size relationships of p^{O*}, p^{TS*}, and p^{TB*} through judge the signal of $p^{TS*}-p^{O*}$, $p^{TB*}-p^{O*}$, and $p^{TB*}-p^{TS*}$. The process is similar with to that of Theorem 1. Note that,

$$\alpha' = 1 - ((-T^3 c_m{}^2 \upsilon^2 F\theta^2 + 2\beta T c_m \upsilon BF(Tc_m\upsilon + 2\beta B)\theta - \beta\upsilon^2 TB(E(T_B)^2\beta^2 - E(T_B)T\beta^2 - T^2)c_m{}^2 + 2\beta\upsilon TB(E(N)Gc_r - FJ)c_m - \beta B(E(N)^2 Gc_r{}^2 - 2E(N)FGc_r + F(E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B)T\beta - 3T^2\beta + T^2))\upsilon + \beta BFG((c_m\upsilon - 1)^2 T)) / 4BFG\delta\beta),$$

$$\alpha'' = ((\upsilon(\theta-1)c_m - \beta + 1)T + E(N)c_r)^2 / (4T\delta(\beta - 1)) + 1 .$$

Proof of Proposition 2

By judging the signal of $p_A^{O*} - p_A^{TB*}$, $p_A^{O*} - p_A^{TS*}$, $p_A^{TB*} - p_A^{TS*}$, we can get Proposition 2 (a) easily.

By judging the signal of $p_{AB}^{TB*} - (p_A^{TS*} + p_B^{TS*}) = J(c_m T\upsilon(\theta - 1) - J)/G$, we can get Proposition 2 (b).

Proof of Proposition 3

The proof process of Proposition 3 is similar to Theorem 1 and Theorem 2.

Proof of Proposition 4

Firstly, we can judge $D_A^{O*}/D_P^{O*} = (D_A^{TB*} + D_{AB}^{TB*})/D_P^{TB*}$ easily, because $D_A^{O*}/D_P^{O*} - (D_A^{TB*} + D_{AB}^{TB*})/D_P^{TB*} = 0$. Next, the signal of $(D_A^{TS*} + D_{AB}^{TS*})/D_P^{TS*} - D_A^{O*}/D_P^{O*}$, which is equal to $(T\upsilon(\theta F - B\beta)c_m + c_r E(N)B\beta)/(2\beta BF)$, is different due to the value of c_m . If $c_m < \hat{c}_m$, we have $(D_A^{TS*} + D_{AB}^{TS*})/D_P^{TS*} > D_A^{O*}/D_P^{O*} = (D_A^{TB*} + D_{AB}^{TB*})/D_P^{TB*}$; otherwise, we have $D_A^{O*}/D_P^{O*} = (D_A^{TB*} + D_{AB}^{TB*})/D_P^{TB*} > (D_A^{TS*} + D_{AB}^{TS*})/D_P^{TS*}$. Note that $\hat{c}_m = -c_r E(N)B\beta/(T\upsilon(\theta F - B\beta))$.

As for the size relationships of the proportion of the FREW in cases TS and TB, we have $(D_B^{TS*} + D_{AB}^{TS*})/D_P^{TS*} - D_{AB}^{TB*}/D_P^{TB*} = (T\upsilon(\theta F - B\beta)c_m + c_r E(N)B\beta)/2FT(\beta - 1)$. Then, we can get Proposition 4 (b) easily.

Proof of Theorem 1

At first, we calculate $\Pi_F^{TS*} - \Pi_F^{O*} = g(\alpha)f(\alpha) - C_E$, where

$$g(\alpha) = ((-4\beta BFGA - T^3 c_m{}^2\upsilon^2 F\theta^2 + 2\beta Tc_m\upsilon BF(Tc_m\upsilon + 2\beta B)\theta - \beta\upsilon^2 TB(E(T_B)\beta^2(E(T_B) - T) - T^2)c_m{}^2 + 2\beta\upsilon TB(E(N)Gc_r - FJ)c_m - \beta B(E(N)^2 Gc_r{}^2 - 2E(N)FGc_r + F(E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B)T\beta - 3T^2\beta + T^2)))/(8\beta BFG)) - (((c_m\upsilon - 1)^2 T + 4A')/8),$$

$$f(\alpha) = ((-4\beta BFGA - T^3 c_m{}^2\upsilon^2 F\theta^2 + 2\beta Tc_m\upsilon BF(Tc_m\upsilon + 2\beta B)\theta - \beta\upsilon^2 TB(E(T_B)^2\beta^2 - E(T_B)T\beta^2 - T^2) c_m{}^2 + 2\beta\upsilon TB(E(N)Gc_r - FJ)c_m - \beta B(E(N)^2 Gc_r{}^2 - 2E(N)FGc_r + F(E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B) T\beta - 3T^2\beta + T^2)))/(8\beta BFG)) + (((c_m\upsilon - 1)^2 T + 4A')/8).$$

According to the conditions to ensure the existence of optimal solutions, we can get the lower limits of α, that is $\underline{\alpha}^O$ and $\underline{\alpha}^{TS}$. Then, $\underline{\alpha} = \max\{\underline{\alpha}^{TS}, \underline{\alpha}^O\}$. According to $g(\underline{\alpha}) < 0$, we have $g(\alpha) < 0$ within the range. As for $f(\alpha)$, the coefficient of α is negative. If $\alpha \le \tilde{\alpha}$, we have $f(\alpha) \ge 0$. At that time, $g(\alpha)f(\alpha) \le 0$. Thus, $\Pi_F^{TS*} - \Pi_F^{O*} \le 0$. Only when $\alpha > \tilde{\alpha}$, $g(\alpha)f(\alpha) > 0$. We can get Theorem 1 (a).

Then, based on $\Pi_F^{TB*} - \Pi_F^{O*} = h(c_m)t(\alpha) - C_E$, where

$$h(c_m) = (4T(\beta - 1)A + T^2 c_m{}^2\theta^2\upsilon^2 - 2c_m((c_m\upsilon + \beta - 1)T - E(N)c_r)\upsilon T\theta + \beta(c_m{}^2\upsilon^2 + \beta - 1)T^2 - 2E(N)c_r (c_m\upsilon + \beta - 1)T + E(N)^2 c_r{}^2)/(8(\beta - 1)T) - (((c_m\upsilon - 1)^2 T - 4c + 4\delta)/8),$$

$$t(\alpha) = (4T(\beta - 1)A + T^2 c_m{}^2\theta^2\upsilon^2 - 2c_m((c_m\upsilon + \beta - 1)T - E(N)c_r)\upsilon T\theta + \beta(c_m{}^2\upsilon^2 + \beta - 1)T^2 - 2E(N)c_r (c_m\upsilon + \beta - 1)T + E(N)^2 c_r{}^2)/(8(\beta - 1)T) + (((c_m\upsilon - 1)^2 T - 4c + 4\delta)/8).$$

Similarly, we substitute $\underline{\alpha}$ which is $\max\{\underline{\alpha}^{TB}, \underline{\alpha}^O\}$ into $t(\alpha)$. According to $t(\underline{\alpha}) > 0$, we have $t(\alpha) > 0$ within the available range. Then, the root formula of the numerator of $h(c_m)$ shows that it is greater than or equal to 0. Thus, we have $h(c_m)t(\alpha) > 0$. It is easy to obtain theorem 1 (b). Note that,

$$\tilde{\alpha}=((-T^3c_m{}^2\upsilon^2F\theta^2+2\beta Tc_m\upsilon BF(Tc_m\upsilon+2\beta B)\theta-\beta\upsilon^2TB(E(T_B)^2\beta^2-E(T_B)T\beta^2-T^2)c_m{}^2+2\beta\upsilon TB(E(N)Gc_r-FJ)c_m$$
$$-\beta B(E(N)^2Gc_r{}^2-2E(N)FGc_r+F(E(T_B)^2\beta^2-E(T_B)T\beta^2+2E(T_B)T\beta-3T^2\beta+T^2))\upsilon+\beta BFG((c_m\upsilon-1)^2T))/4BFG\delta\beta)+1$$

,

$$C_E^I=((-4\beta BFGA-T^3c_m{}^2\upsilon^2F\theta^2+2\beta Tc_m\upsilon BF(Tc_m\upsilon+2\beta B)\theta-\beta\upsilon^2TB(E(T_B)^2\beta^2-E(T_B)T\beta^2-T^2)$$
$$c_m{}^2+2\beta\upsilon TB(E(N)Gc_r-FJ)c_m-\beta B(E(N)^2Gc_r{}^2-2E(N)FGc_r+F(E(T_B)^2\beta^2-E(T_B)T\beta^2+2E(T_B)\ ,$$
$$T\beta-3T^2\beta+T^2)))^2/(64\beta^2B^2F^2G^2))-((c_m\upsilon-1)^2T-4c+4\delta)^2/64$$
$$C_E^{II}=(4T(\beta-1)A+T^2c_m{}^2\theta^2\upsilon^2-2c_m((c_m\upsilon+\beta-1)T-E(N)c_r)\upsilon T\theta+\beta(c_m{}^2\upsilon^2+\beta-1)T^2-2E(N)c_r$$
$$(c_m\upsilon+\beta-1)T+E(N)^2c_r{}^2)^2/(64(\beta-1)^2T^2)-(((c_m\upsilon-1)^2T-4c+4\delta)^2/64)\ .$$

Proof of Theorem 2

Firstly, we obtain $\Pi_F^{TS*}-\Pi_F^{TB*}=w(c_m)k(\alpha)$, where

$$w(c_m)=((-4\beta BFGA-T^3c_m{}^2\upsilon^2F\theta^2+2\beta Tc_m\upsilon BF(Tc_m\upsilon+2\beta B)\theta-\beta\upsilon^2TB(E(T_B)^2\beta^2-E(T_B)T\beta^2$$
$$-T^2)c_m{}^2+2\beta\upsilon TB(E(N)Gc_r-FJ)c_m-\beta B(E(N)^2Gc_r{}^2-2E(N)FGc_r+F(E(T_B)^2\beta^2-E(T_B)T\beta^2$$
$$+2E(T_B)T\beta-3T^2\beta+T^2)))/(8\beta BFG))+(4T(\beta-1)A+T^2c_m{}^2\theta^2\upsilon^2-2c_m((c_m\upsilon+\beta-1)T-E(N)c_r)\ ,$$
$$\upsilon T\theta+\beta(c_m{}^2\upsilon^2+\beta-1)T^2-2E(N)c_r(c_m\upsilon+\beta-1)T+E(N)^2c_r{}^2)/(8(\beta-1)T)$$
$$k(\alpha)=((-4\beta BFGA-T^3c_m{}^2\upsilon^2F\theta^2+2\beta Tc_m\upsilon BF(Tc_m\upsilon+2\beta B)\theta-\beta\upsilon^2TB(E(T_B)^2\beta^2-E(T_B)T\beta^2$$
$$-T^2)c_m{}^2+2\beta\upsilon TB(E(N)Gc_r-FJ)c_m-\beta B(E(N)^2Gc_r{}^2-2E(N)FGc_r+F(E(T_B)^2\beta^2-E(T_B)T\beta^2$$
$$+2E(T_B)T\beta-3T^2\beta+T^2)))/(8\beta BFG))-(4T(\beta-1)A+T^2c_m{}^2\theta^2\upsilon^2-2c_m((c_m\upsilon+\beta-1)T-E(N)c_r)\ .$$
$$\upsilon T\theta+\beta(c_m{}^2\upsilon^2+\beta-1)T^2-2E(N)c_r(c_m\upsilon+\beta-1)T+E(N)^2c_r{}^2)/(8(\beta-1)T)$$

Similarly, we substitute $\underline{\alpha}$ which is equal to $\max\{\underline{\alpha}^{TB},\underline{\alpha}^{TS}\}$ into $k(\alpha)$ and then have $k(\alpha)<0$. As for $w(c_m)$, it is the quadratic equation of c_m. Then we can obtain the lower limit of c_m from the existence of the demand of each extended warranty, where $\underline{c}_m=(-B\beta(GE(N)c_r+FJ))/T^2\upsilon(B\beta^2-F\theta)$. We can easily know $w(\underline{c}_m)>0$. Next, we judge the coefficient of quadratic term, that is $c_m{}^2$. If the coefficient of $c_m{}^2$ is greater than 0, that is, $\theta<\tilde{\theta}$, we have $w(c_m)>0$ within the available range. If the coefficient of $c_m{}^2$ is smaller than 0, the signal of $w(c_m)$ will change with the increase of c_m. Hence, Theorem 2 (a) and (b) can be drawn. Note that,

$$\tilde{\theta}=(BE(T_B)(2T-E(T_B))\beta^3-B(2T-E(T_B))T\beta^2+\sqrt{B(\beta-1)^3\beta FGT^3})/(F(TF-B^2\beta^2))\ ,$$
$$\tilde{c}_m=(2\beta\upsilon TB((E(N)Gc_r+TJ(\beta-1))F\theta-E(N)\beta BGc_r-T(\beta-1)FJ)+2\sqrt{\upsilon^2BFGT^3\beta(\beta-1)H})$$
$$/(2\upsilon^2T^2(\theta^2F(TF-B^2\beta^2)+2\theta\beta^2BG(B-T)-B\beta((E(T_B)^2-3E(T_B)+T^2)\beta^2+2T^2\beta-T^2)))\ .$$

Proof of Proposition 5

Note that,

$$\tilde{c}_m=(2\beta\zeta TB((E(N)Gc_r+TJ(\beta-1))F\theta-E(N)\beta BGc_r-T(\beta-1)FJ)+2\sqrt{\upsilon^2BFGT^3\beta(\beta-1)H})$$
$$/(2\upsilon^2T^2(\theta^2F(TF-B^2\beta^2)+2\theta\beta^2BG(B-T)-B\beta((E(T_B)^2-3E(T_B)+T^2)\beta^2+2T^2\beta-T^2)))\ .$$

We set $P=2\beta\nu TB((E(N)Gc_r+TJ(\beta-1))F\theta-E(N)\beta BGc_r-T(\beta-1)FJ)$, and then find the partial derivative of P regarding to c_r and the partial derivative of H regarding to c_r, that is $\partial P/\partial c_r$ and $\partial\sqrt{H}/\partial c_r$. Note that, $\partial P/\partial c_r=-B\upsilon\beta E(N)T(\mathrm{B}\beta-F\theta)\mathrm{G}>0$, $\partial\sqrt{H}/\partial c_r=E(N)B\beta(\theta-1)>0$. Thus, we have $\partial\tilde{c}_m/\partial c_r>0$.

Proof of Proposition 6

According to the expression and meaning of $E(\mathrm{T_B})$ and $E(N)$. We can get Proposition 6 easily.

Proof of Proposition 7

$\partial p_A^{O*} / \partial \upsilon > 0$ can be easily drawn. Then, to judge the signal of $\partial p_A^{TS*} / \partial \upsilon$, we firstly differentiate p_A^{TS*} to $E(\mathrm{T_B})$ and to $E(N)$. Then combining Proposition 6, we can get $\partial p_A^{TS*} / \partial \upsilon > 0$. As for $\partial p_A^{TB*} / \partial \upsilon$, $\partial p_{AB}^{TB*} / \partial \upsilon$, $\partial p_A^{TS*} / \partial N$, and $\partial p_{AB}^{TB*} / \partial N$, we can obtain their signal through the similar process. Hence, Proposition 7 can be drawn.

Proof of Theorem 3

The proof process is similar to theorem 1.

Proof of Theorem 4

The proof process is similar to theorem 2.

Proof of Theorem 5

The proof process is similar to theorem 1. Note that,

$$U_F^{RTB*} = (2T(\beta-1)(2-\eta)A + T^2 c_m{}^2\theta^2\upsilon^2 - 2c_m((c_m\upsilon+\beta-1)T - E(N)c_r)\upsilon T\theta + \beta(c_m{}^2\upsilon^2+\beta-1)T^2 \\ -2E(N)c_r(c_m\upsilon+\beta-1)T + E(N)^2 c_r{}^2)^2 / (8(\beta-1)^2(2-\eta)^3 T^2) - C_E,$$

$$U_F^{RTS*} = ((2(2-\eta)\beta ABFG + T^3 c_m{}^2\upsilon^2 F\theta^2 + 2\beta Tc_m\upsilon BF(J\eta - 2\beta B - Tc_m\upsilon) + \beta\upsilon^2 TB(E(T_B)^2\beta^2 - \\ E(T_B)T\beta^2 - T^2)c_m{}^2 - 2\beta\upsilon TB(E(N)Gc_r + (\eta-1)FJ)c_m + E(N)^2\beta BGc_r{}^2 - 2\beta BGFE(N)c_r + \beta BF \\ (J^2\eta^2 - 2J^2\eta + E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B)T\beta - 3T^2\beta + T^2)))^2 / (8\beta^2 B^2 F^2 G^2(2-\eta)^3) - C_E,$$

$$U_F^{RO*} = ((c_m\upsilon-1)^2 T + 2(2-\eta)(\delta-c))^2 / (8(2-\eta)^3),$$

$$C_E^{R\prime} = ((2(2-\eta)\beta ABFG + T^3 c_m{}^2\upsilon^2 F\theta^2 + 2\beta Tc_m\upsilon BF(J\eta - 2\beta B - Tc_m\upsilon) + \beta\upsilon^2 TB(E(T_B)^2\beta^2 - \\ E(T_B)T\beta^2 - T^2)c_m{}^2 - 2\beta\upsilon TB(E(N)Gc_r + (\eta-1)FJ)c_m + E(N)^2\beta BGc_r{}^2 - 2\beta BGFE(N)c_r + \beta BF \\ (J^2\eta^2 - 2J^2\eta + E(T_B)^2\beta^2 - E(T_B)T\beta^2 + 2E(T_B)T\beta - 3T^2\beta + T^2)))^2 / (8\beta^2 B^2 F^2 G^2(2-\eta)^3) - ((c_m\upsilon \\ -1)^2 T + 2(2-\eta)(\delta-c))^2 / (8(2-\eta)^3),$$

$$C_E^{R\prime\prime} = (2T(\beta-1)(2-\eta)A + T^2 c_m{}^2\theta^2\upsilon^2 - 2c_m((c_m\upsilon+\beta-1)T - E(N)c_r)\upsilon T\theta + \beta(c_m{}^2\upsilon^2+\beta-1)T^2 \\ -2E(N)c_r(c_m\upsilon+\beta-1)T + E(N)^2 c_r{}^2)^2 / (8(\beta-1)^2(2-\eta)^3 T^2) - ((c_m\upsilon-1)^2 T + 2(2-\eta)(\delta-c))^2 \\ /(8(2-\eta)^3),$$

$$\tilde{\alpha}^R = ((-T^3 c_m{}^2\upsilon^2 F\theta^2 - 2\beta Tc_m\upsilon BF(J\eta - 2\beta B - Tc_m\upsilon)\theta - \beta\upsilon^2 T^2 B(2E(T_B)\beta - T\beta - 2T)c_m{}^2 \\ +2\beta\upsilon TB(E(N)Gc_r + F(J\eta - 2\beta B))c_m - \beta B(E(N)^2 Gc_r{}^2 - 2E(N)FGc_r + F(J^2(\eta^2 - 2\eta) \\ +E(T_B)^2\beta^2 - E(T_B)T\beta^2 + E(T_B)T\beta - 2T^2\beta + 2T^2)) / (2(2-\eta)\delta BFG\beta)) + 1.$$

Proof of Theorem 6

The proof process is similar to theorem 2. Note that,

$$\tilde{c}_m^R = (2\sqrt{\upsilon^2 BFGT^3\beta(\beta-1)H^R} + 2\beta\upsilon TB((E(N)Gc_r + TJ(\beta-1)(1-\eta))F\theta - E(N)\beta BGc_r \\ -T(\beta-1)FJ(1-\eta))) / (2\upsilon^2 T^2(\theta^2 F(TF - B^2\beta^2) + 2\theta\beta^2 BG(B-T) - B\beta((E(T_B)^2 - 3E(T_B) \\ +T^2)\beta^2 + 2T^2\beta - T^2))).$$