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Methods of influence coefficients to evaluate stress and deviation distributions of parts under operating conditions - A review

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A B S T R A C T

The most used approach to solve tolerance analysis problems for flexible products is the method of influence coefficients that combines the finite element analysis with statistical analysis in order to establish a relationship between the product deviation and part deviation and to foresee the statistical distribution of stresses. The key of this relationship is the sensitivity matrix for the deviations and stresses that can be evaluated by different methods of influence coefficients. Therefore, the aim of this work is to make a review of these methods applying them to evaluate on some single parts the statistical distribution of deviations and stresses due to operating conditions, i.e. due to the displacements applied to the part during its working.

1. Introduction

All products have geometrical and dimensional deviations from the nominal geometry. These deviations can lead to problems during the assembly because they propagate component-to-component by influencing the perceived quality of the products and their intended functionality. Moreover, the stresses introduced as a consequence of non-nominal assembly should also be kept below critical limits during the conditions of use. In the design of a product, the numerical simulation is one of the most important activity of geometry assurance (Söderberg et al. 2016) in order to understand how the deviations propagate. A large number of models for tolerance analysis of rigid assemblies have been proposed (Cao et al., 2018); however, they are inadequate when components exhibit a non-rigid behaviour. For this reason, finite element analysis (FEA)-based methods are necessary.

Direct Monte Carlo (DMC) method can be used to solve tolerance analysis problems in combination with the finite element analysis (FEA), but the evaluation of product results can be very time consuming. To overcome this problem, the method of influence coefficient (MIC) is commonly used to avoid solving the FEA at each Monte Carlo iteration (Liu and Hu 1997). This is due to the fact that using MIC is possible to calculate the sensitivity matrix. The sensitivity matrix tells the designer which assembly parameter variations has the greatest effect on the critical assembly geometry. In other words, once the geometries, materials and boundary conditions are defined (how the parts to be assembled are

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constrained, how the parts are assembled, etc...), the sensitivity matrix is unique; therefore, it is possible to foresee the deviations of the product as the single parts deviate from their nominal geometry.

Over the years, the MIC was evolved; in fact, it was extended to multi-assembly processes by considering deviations due to fixturing and tooling equipment (Camelio et al., 2003); the geometric covariance was introduced to consider the deviations as dependent with each other (Camelio et al., 2004); different solutions were proposed to manage the non-linearity due to the contact between the bodies (Dahlström and Lindkvist 2006; Ungemach and Mantwill 2008; Lindau et al., 2016); the thermal expansion together with geometrical variation and an integration of manufacturing process simulations of composite materials into the field of tolerance analysis were taken into account (Lorin et al., 2010, 2013; Jareteg et al. 2016). Moreover, a further reduction of the simulation time for the MIC is a current topic of study. By using the method proposed in (Lindau et al. 2014), it was possible, once known the displacements of a subset of nodes, to estimate their contribution on the rest of the nodes describing the entire product, ensuring shorter simulation times. By using the Sherman-Morrison-Woodbury formula in the MIC, it was possible to save further simulation time in order to solve problems of non-rigid assemblies (Lorin et al. 2017). By using the super-element method, it was possible to solve tolerance analysis problems with a more efficient evaluation of the stiffness matrix of the bodies involved in the assembly (Corrado et al., 2019).

The aim of this work is to make a review on how to evaluate the sensitivity matrix for deviations and stresses using different methods of influence coefficients. The sensitivity matrices can be evaluated through the matrix of influence coefficients that can be determined by the unit force method (Liu and Hu 1997), the unit deviation method (Sellem and Rivière 1998; Gerbino et al., 2008) or the super-elements method (Corrado et al. 2019). The methods of influence coefficients were created to solve tolerance analysis problems for flexible products. In this work, however, these methods were used on single parts in order to evaluate on them the statistical distribution of deviations and stresses due to operating conditions, such as the tensile and compressive conditions simulated by an imposed and variable displacement. The operating conditions are of great interest for the tolerance analysis problems (Armillotta and Semeraro 2013; Corrado et al. 2017, 2018; Zeng and Rao 2019). The present work is organised as follows: in Section 2, some methods to evaluate the sensitivity matrix for the deviations and stresses are discussed. In Section 0, two case studies are presented, solved and compared in terms of simulation time and result accuracy. A summary of advantages and disadvantages of the three different methods is discussed together with a presentation of suggestions to improve them. Finally, the conclusions are presented.

2. Methods of influence coefficients for deviation and stress simulations

The most used approach to solve tolerance analysis problems is the method of influence coefficients (MIC) (Liu & Hu, 1997). The MIC combines finite element analysis (FEA) with statistical analysis and it establishes a relationship between the product deviation $\{V\}_a$ and part deviation $\{V\}_p$. This is the key of the method and it is described by this equation:

$$\{V\}_a = [K]_a^{-1} \cdot [K]_p \cdot \{V\}_p = [S_\delta] \cdot \{V\}_p \tag{1}$$

where $[S_\delta]$ is the sensitivity matrix, $[K]_a^{-1}$ and $[K]_p$ are the inverse of stiffness matrix of the product and the stiffness matrix of the parts to be assembled respectively. The part deviation $\{V\}_p$ is a *n* x 1 vector and the product deviation $\{V\}_a$ is a 3*m* x 1 vector, where *n* is equal to the number of nodes with input deviations, *m* is equal to the total number of the geometry nodes and 3 are the main directions of deviation. Generally, n is less than m, or at most equal. Therefore, $[K]_p$ is a n x n matrix, $[S_\delta]$ and $[K]_a^{-1}$ are $3m$ x n matrices.

Generally, the part deviation $\{V\}_p$ is statistically defined in terms of mean deviation, variance or covariance. Thus, Eq. (1) can be expressed in terms of mean deviations:

$$\{\mu\}_a = [K]_a^{-1} \cdot [K]_p \cdot \{V\}_p = [S_\delta] \cdot \{\mu\}_p \tag{2}$$

and variance, if the deviations of parts are statistically independent:

$$\{\sigma^2\}_a = \left[[K]_a^{-1} \cdot [K]_p\right]^2 \cdot \{\sigma^2\}_p = [S_\delta^2] \cdot \{\sigma^2\}_p \tag{3}$$

where $\{\mu\}_p$ and $\{\mu\}_a$ are the mean deviation vectors of parts to be assembled and product respectively; $\{\sigma^2\}_p$ and $\{\sigma^2\}_a$ are the variance vectors of parts to be assembled and the product respectively. If the deviations of parts to be assembled are statistically dependent, the covariance matrix of product is evaluated as:

$$[\Sigma]_a = \left[[K]_a^{-1} \cdot [K]_p\right] \cdot [\Sigma]_p \cdot \left[[K]_a^{-1} \cdot [K]_p\right]^T = [S_\delta] \cdot [\Sigma]_p \cdot [S_\delta]^T \tag{4}$$

where $[\Sigma]_p$ and $[\Sigma]_a$ are is the covariance matrices of parts to be assembled and the product respectively. The mean deviation vector $\{\mu\}_p$ and the variance vector $\{\sigma^2\}_p$ have the same dimensions of $\{V\}_p$. The mean deviation vector $\{\mu\}_a$ and the variance vector $\{\sigma^2\}_a$ have the same dimensions of $\{V\}_a$. The covariance matrix of parts $[\Sigma]_p$ is a n x n matrix, the covariance matrix $[\Sigma]_a$ is a $3m \times 3m$ matrix.

As said in the Introduction section, the method of influence coefficients was created to solve tolerance analysis problems for flexible products. However, in this work, this method was used on single parts and not on products. So, there is no substantial difference between parts and product because they represent the same thing since there is no assembly operation.

In the following paragraphs, this distinction will continue to be made between parts and products to better distinguish the variables to be calculated and the steps in which to calculate them.

2.1 Unit force method and sensitivity matrix evaluation

When the method of influence coefficients was proposed, the FEA software did not provide the stiffness matrices explicitly (Liu & Hu, 1997), so the sensitivity matrix could not be evaluated. The method of influence coefficients evaluates the sensitivity matrix by two steps. The first step foresees a unit force applied at the i-th node of part geometry where there is a deviation (i = 1 to n nodes). FEA software calculates the displacement under the i-th unit force in the n nodes. These displacements are recorded into a n x n matrix $[C]$, called influence coefficients matrix, and its inverse represents the stiffness matrix of the parts $[K]_p$. Therefore, n FEA simulations are required to evaluate $[K]_p$. The second step foresees that the i-th column of matrix $[K]_p$ is treated as the forces to be applied on the nodes affected by deviations. FEA software calculates again the displacements that are recorded in the matrix $[S_\delta]$. Therefore, further n FEA simulations are required to evaluate the sensitivity matrix.

2.2 Unit displacement method and sensitivity matrix evaluation

Another way to calculate the sensitivity matrix was proposed in (Sellem and Rivière 1998; Gerbino et al., 2008). The authors proposed to apply unit displacements to nodes of parts where there are deviations and to evaluate the stack-up of deviations in the product. The unit displacement method was also used to determine the assembly sensitivity matrix that relates product deviations to fixture variations (Söderberg et al., 2006). In this method, the influence coefficients matrix is defined as a *n* x *n* matrix, where each element represents the reaction force at the *i*-th node due to the unit displacement that stands for a deviation on part geometries. In other words, the influence coefficients matrix corresponds to the stiffness matrix of parts defined as $[K]_p$ previously. Also in this case, the method foresees two steps. In the first step, a unit displacement is applied at the *i*-th node of part geometries where there is a deviation (i = 1 to n). FEA software calculates the reaction forces under the *i*-th unit displacement in the *n* nodes. The reaction forces are recorded into a $n \times n$ matrix called influence coefficients matrix. Therefore, *n* FEA simulations are required to evaluate that matrix. The second step is identical to the second step defined above. Therefore, further *n* FEA simulations are required to evaluate the sensitivity matrix. The advantage of this method is that it is not necessary to invert the matrix $[C]$ in the stiffness matrix because the stiffness matrix is obtained and used to evaluate the sensitivity matrix $[S_\delta]$ directly.

2.3 Super-elements method and sensitivity matrix evaluation

The Cassino Unified Tolerance Analysis (CaUTA) is a tool to solve tolerance analysis problems (Corrado and Polini 2019) and it was presented to keep together the tolerance analysis on rigid and flexible products in order that all functions, which characterized the two kinds of analysis, could be available in the same tool and at the same time. The CaUTA tool can solve tolerance analysis problems of flexible assemblies by using the (DMC) method or the method of influence coefficients; with this last method, $[K]_p$ can be evaluated in three ways: as described in §2.1 and §2.2, and explicitly as explained in the following. In fact, nowadays, the FEA software calculates the stiffness matrix $[K]_p$ explicitly, unlike what happened in the past. In particular, the CaUTA tool uses MSC Marc Mentat®, as FEA software, that evaluates the stiffness matrix by super-elements method (Corrado et al., 2019). The super-elements method reduces the degrees of freedom (DOFs) of the global stiffness matrix to only those DOFs that exist on specific nodes. The DOFs of a super-element are classified as internal and boundary. The internal DOFs are those that are not connected to the DOFs of another super-element, and the nodes with internal DOFs are called internal nodes. The boundary DOFs are those that are connected to at least one other super-element; the nodes with boundary DOFs are called boundary nodes. The reduction process of DOFs is called static condensation, or Guyan condensation (Qu 2004), and to carry out the reduction it is necessary to start from the general form of the global stiffness matrix:

$$[K]\cdot\{\delta\}=\{f\} \tag{5}$$

that can be partitioned as:

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix}\cdot\begin{Bmatrix} \delta_b \\ \delta_i \end{Bmatrix}=\begin{Bmatrix} f_b \\ f_i \end{Bmatrix} \tag{6}$$

where the sub-vectors $\{\delta_b\}$ and $\{\delta_i\}$ collect the boundary and internal DOFs respectively. The second equation of expression (6) can be written as:

$$K_{ib}\cdot\delta_b+K_{ii}\cdot\delta_i=f_i \tag{7}$$

The Eq. (7) can be solved for internal DOFs if $\left[K_{ii}\right]$ is not-singular:

$$\delta_i = K_{ii}^{-1} \cdot \left(f_i - K_{ib} \cdot \delta_b \right) \tag{8}$$

The condensed stiffness equation is evaluated by replacing $\left\{\delta_i\right\}$ into the first equation of expression (6):

$$\tilde{K}_{bb} \cdot \delta_b = \tilde{f}_b \quad \text{where} \quad \begin{cases} \tilde{K}_{bb} = K_{bb} - K_{bi} \cdot K_{ii}^{-1} \cdot K_{ib} \\ \tilde{f}_b = f_b - K_{bi} \cdot K_{ii}^{-1} \cdot f_i \end{cases} \tag{9}$$

and $\tilde{K}_{bb}$ is called condensed stiffness matrix and represents the stiffness matrix $\left[K\right]_p$ with a reduction of the DOFs to only those DOFs that exist on specific nodes, i.e. the nodes where there are deviations. This condensation helps to reduce the dimensions of stiffness matrix without loss of accuracy, as explained in (Schrefler 1988). Moreover, the simulation time decreases because only 1 FEA simulation is necessary to evaluate $\left[K\right]_p$ in MSC Marc Mentat®. Once evaluated $\left[K\right]_p$, the CaUTA tool calculates the sensitivity matrix $\left[S_\delta\right]$, as described in §2.1 and §2.2. The *i*-th column of matrix $\left[K\right]_p$ is applied on the nodes affected by deviations as a force and FEA software calculates the displacements that are recorded in the matrix $\left[S_\delta\right]$. Therefore, further *n* FEA simulations are required to evaluate the sensitivity matrix.

2.4 The sensitivity matrix for stresses

The described methods allow calculating the sensitivity matrix for the dimensional and geometric deviations. The statistical stress distribution, due to dimensional and geometrical deviations of parts to be assembled, is not frequently considered in the assembly, even if it is an important aspect in the product design. Anyway, in the last years, some statistical works were presented (Lorin et al., 2014; Söderberg et al., 2015) where the method of influence coefficients was used to simulate the statistical distribution of stresses for a single part and a product of composite parts. In particular, the unit displacement (or unit disturbance) method was used to evaluate the sensitivity matrices for the deviations and stresses. However, the unit force method and the super-element method can also be used to evaluate the sensitivity matrix for stress in addition to the sensitivity matrix of the deviations. In fact, once evaluated $\left[K\right]_p$ with one of the three methods, if the *i*-th column of matrix $\left[K\right]_p$ is treated as the forces to be applied on the nodes affected by deviations, FEA software can calculate the displacements and the stresses (in different directions) on the product due to those forces. In this way, the displacements are recorded in a sensitivity matrix $\left[S_\delta\right]$ for deviations and the stresses are recorded in a sensitivity matrix $\left[S_{\psi,\tau}\right]$ for stresses simultaneously by running only *n* FEA simulations; where ψ and τ represent the normal and the shear stress contribution respectively. At this point, the Eqs. (2-4) can be redefined as follows:

$$\begin{aligned} \left\{\mu_\delta\right\}_a &= \left[S_\delta\right] \cdot \left\{\mu\right\}_p \\ \left\{\mu_{\psi,\tau}\right\}_a &= \left[S_{\psi,\tau}\right] \cdot \left\{\mu\right\}_p \end{aligned} \tag{10}$$

$$\begin{aligned} \left\{\sigma_\delta^2\right\}_a &= \left[S_\delta^2\right] \cdot \left\{\sigma^2\right\}_p \\ \left\{\sigma_{\psi,\tau}^2\right\}_a &= \left[S_{\psi,\tau}^2\right] \cdot \left\{\sigma^2\right\}_p \end{aligned} \tag{11}$$

$$\begin{aligned} \left[\Sigma_\delta\right]_a &= \left[S_\delta\right] \cdot \left[\Sigma\right]_p \cdot \left[S_\delta\right]^T \\ \left[\Sigma_{\psi,\tau}\right]_a &= \left[S_{\psi,\tau}\right] \cdot \left[\Sigma\right]_p \cdot \left[S_{\psi,\tau}\right]^T \end{aligned} \tag{12}$$

where $\{\mu_\delta\}_a$, $\{\sigma_\delta^2\}_a$ and $[\Sigma_\delta]_a$ are the mean vector, the variance vector and the covariance matrix of the product in terms of deviations; $\{\mu_{\psi,\tau}\}_a$, $\{\sigma_{\psi,\tau}^2\}_a$ and $[\Sigma_{\psi,\tau}]_a$ are the mean vector, the variance vector and the covariance matrix of the product in terms of stresses.

The sensitivity matrix $[S_\delta]$ is a (3*m* x *n*) matrix with *m* equal to total number of geometry nodes, *n* equal to number of nodes with input deviations and 3 are the directions of deviation. The sensitivity matrix $[S_{\psi,\tau}]$ is a (6*m* x *n*) matrix because 6 are the components of stress. Therefore, for example, the resultant mean vector of stresses $\{\mu_{\psi,\tau}\}_a$ can be expressed as:

$$\{\mu_{\psi,\tau}\}_a = \begin{Bmatrix} \overbrace{\psi_x}^{m\ x\ 1} \\ \overbrace{\psi_y}^{m\ x\ 1} \\ \overbrace{\psi_z}^{m\ x\ 1} \\ \overbrace{\tau_{xy}}^{m\ x\ 1} \\ \overbrace{\tau_{xz}}^{m\ x\ 1} \\ \overbrace{\tau_{yz}}^{m\ x\ 1} \end{Bmatrix} \tag{13}$$

where ψ is the normal stress and τ is the shear stress. In this way, the von Mises stress can be evaluated for each *m*-th node by the following equation:

$$\psi_{vM} = \frac{1}{\sqrt{2}} \cdot \sqrt{\left(\psi_x - \psi_y\right)^2 + \left(\psi_y - \psi_z\right)^2 + \left(\psi_z - \psi_x\right)^2 + 6 \cdot \left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)} \tag{14}$$

3. Application cases

The CaUTA tool was used to solve two case studies in the different ways described previously and summarized by a flowchart of Fig. 1. All numerical simulations were run on a computer with an Intel core i7 950 processor running at 3.07 GHz, using 16 GB of RAM, with a mechanical hard drive of 1TB at 5400 rpm, and running Windows 7 Professional.

Case study 1

The first case study foresees the evaluation of stress distribution for a tension strip with opposite semi-circular edge notches (Pilkey and Pilkey 2008). This case study is usually used as example to calculate the stress concentration factors K_{tg} and K_{tn} (stress concentration factor with the nominal stress based on gross area and on net area respectively). In particular, a perfect solution of the stress concentration factors exists for a tension strip with opposite semi-circular edge notches and those values depend on the ratio between the radius of curvature of notch and the width of part according to the chart shown in Fig. 2a.
Due to its symmetry, only a quarter of the tension strip geometry has been considered and the characteristic dimensions, used in this work, are shown in Fig. 2b. The tension strip is in mild steel with a Young's modulus E = 200,000 N/mm^2 and Poisson's ratio υ = 0.3 and it has a thickness (*h*) of 1 mm. The part was discretized with 180 2D elements (QUAD element 3) and constrained as shown in Fig. 2b. In particular, constraints to lock movements along the X and Y directions have been used in order to ensure the symmetry around the Y and X axes respectively.

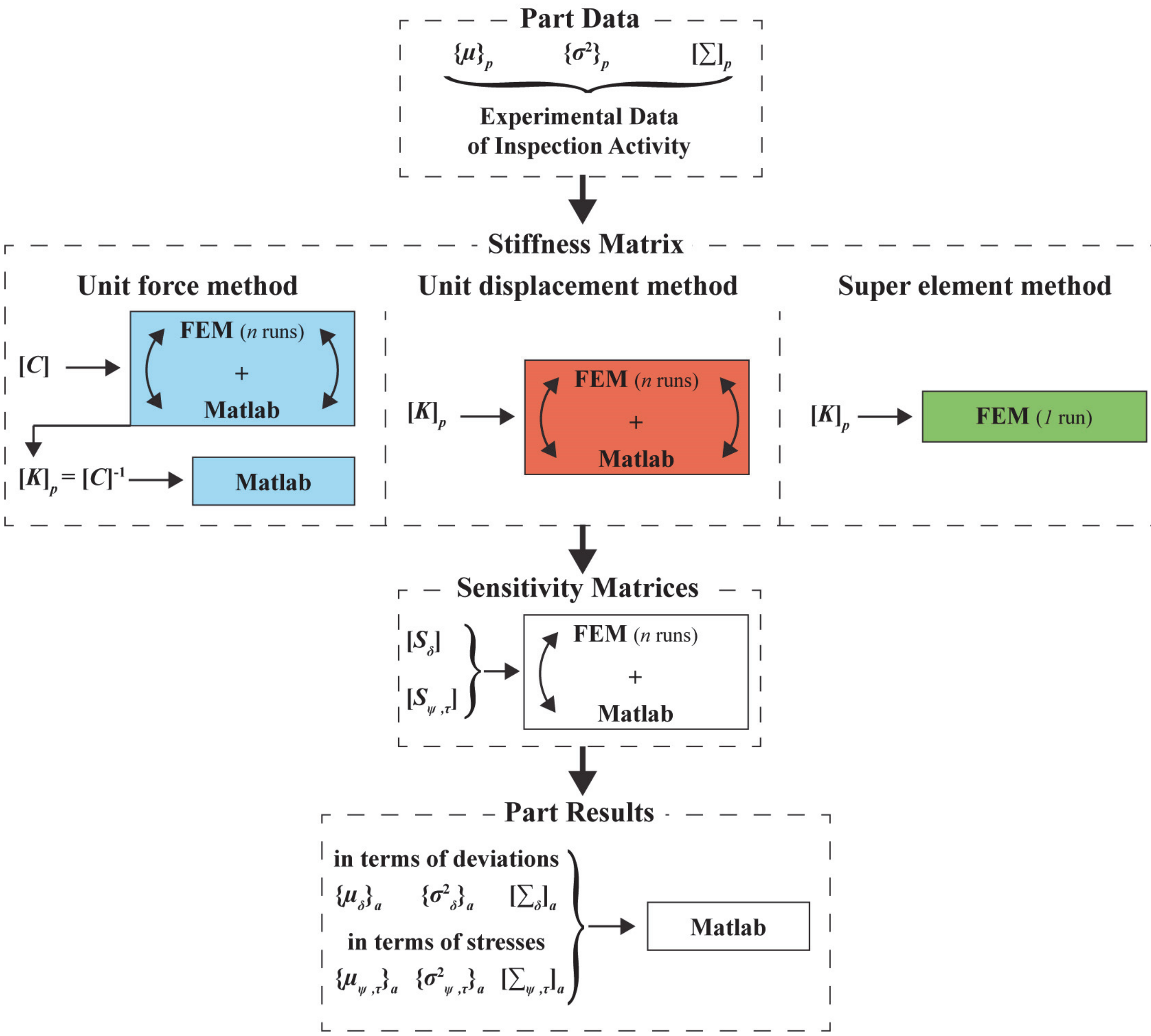

Fig. 1. Schematic differences between methods

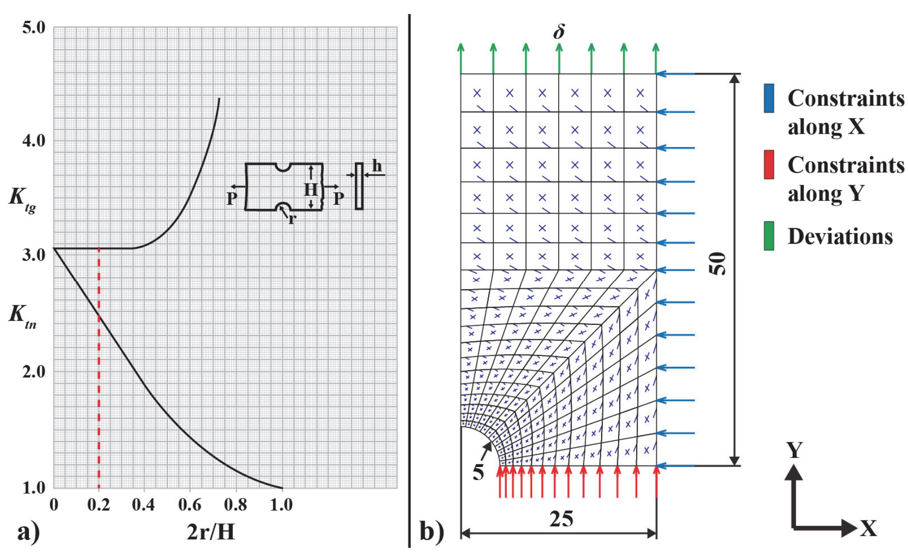

Fig. 2. a) Chart of the stress concentration factors for a tension strip with opposite semi-circular edge notches, b) Details of the first case study (dimensions in mm)

A deviation (δ) on the upper free corner is considered; the mean deviation vector $\{\mu\}_p$ foresees a deviation of 0.025 mm and the variance vector $\{\sigma^2\}_p$ foresees a variance of 0.10 mm in the Y-direction on nodes of the upper free corner. Fig. 3a shows the mean value and standard deviation of deviations on the tension strip along Y-direction obtained from one of the methods to evaluate the sensitivity matrices for deviations. Fig. 3b shows the mean and standard deviation of stresses (ψ_y) on the tension strip along Y-direction obtained from one of the methods to evaluate the sensitivity matrices for stresses. The mean deviations and stresses were evaluated by Eq.(10); the standard deviation was evaluated by taking the square-root of Eq. (11). A maximum error of 0.0001% on both mean and standard deviation values for deviations and stresses was found between one method and another. A significant difference was found only on simulation time. In fact, there was a reduction of simulation time comparing the different methods, as shown in Table 1. As demonstrated in (Corrado et al., 2019), the method presented in §2.3 to evaluate $[K]_p$ allows to almost halve (-45.1% in this case) the simulation time to perform the complete analysis.

Table 1. Comparison between different methods in terms of simulation time for the first case study

Time [s]				**Time reduction [%]**	
	Method			**§0 vs. §0**	**§0 vs. §0**
	§0	§0	§0		
$[K]_p$	75.3	75.3	5.2	0.0	-93.1
$[S_\delta],[S_{\psi,\tau}]$	155.8	158.0	85.5	1.4	-45.1
$\{\mu_\delta\}_a,\{\mu_{\psi,\tau}\}_a$ $\{\sigma_\delta^2\}_a,\{\sigma_{\psi,\tau}^2\}_a$	155.9	158.1	85.6	1.4	-45.1

With the dimensions of the tension strip taken into account, the stress concentration factors K_{tg} and K_{tn} are equal to 3.03 and 2.43, respectively, according to the chart of Fig. 1a. Similar values were also obtained numerically. In fact, stress concentration factors can be evaluated as:

$$K_{tg} = \frac{\psi_{y,\max}}{\psi_y} = 3.07$$
$$K_{tn} = \frac{\psi_{y,\max}}{\psi_{y,nom}} = 2.46 \tag{15}$$

where $\psi_{y,\max}$ is the maximum normal stress on the edge notch, ψ_y is the normal stress on the upper corner and $\psi_{y,nom}$ is the nominal or reference normal stress based on net area of tension strip.

Case study 2

The second case study foresees the evaluation of stress distribution for a helical compression spring (Schmid et al., 2013). This case study is usually used as example to calculate the stiffness matrix. In fact, an analytical solution of the stiffness matrix exists for a helical compression spring:

$$K = \frac{d^4 \cdot G}{8 \cdot N_a \cdot D^3} \tag{16}$$

where d is the wire diameter, D is the mean coil diameter, G is the shear modulus and N_a is the number of active coils. The dimensions of considered helical compression spring are shown in Fig. 4 (d = 5 and D = 2x27.5). The spring is in mild steel with a Young's modulus E = 200,000 MPa, a Poisson's ratio υ = 0.3 and a shear modulus G = 76,923 MPa. The helical spring was discretized with 7680 3D elements (HEX8 element 7) and to simulate the compression of the spring, two surfaces were created and located at the two ends of the spring. These surfaces were defined as rigid bodies and in contact with the spring. In particular, a glue contact is used to connect the spring and two surfaces. The glue condition suppressed all relative motions between the spring and two surfaces through tying applying them to all displacement degrees-of-freedom of the nodes into contact; so, there is no relative tangential motion between the spring and two surfaces.

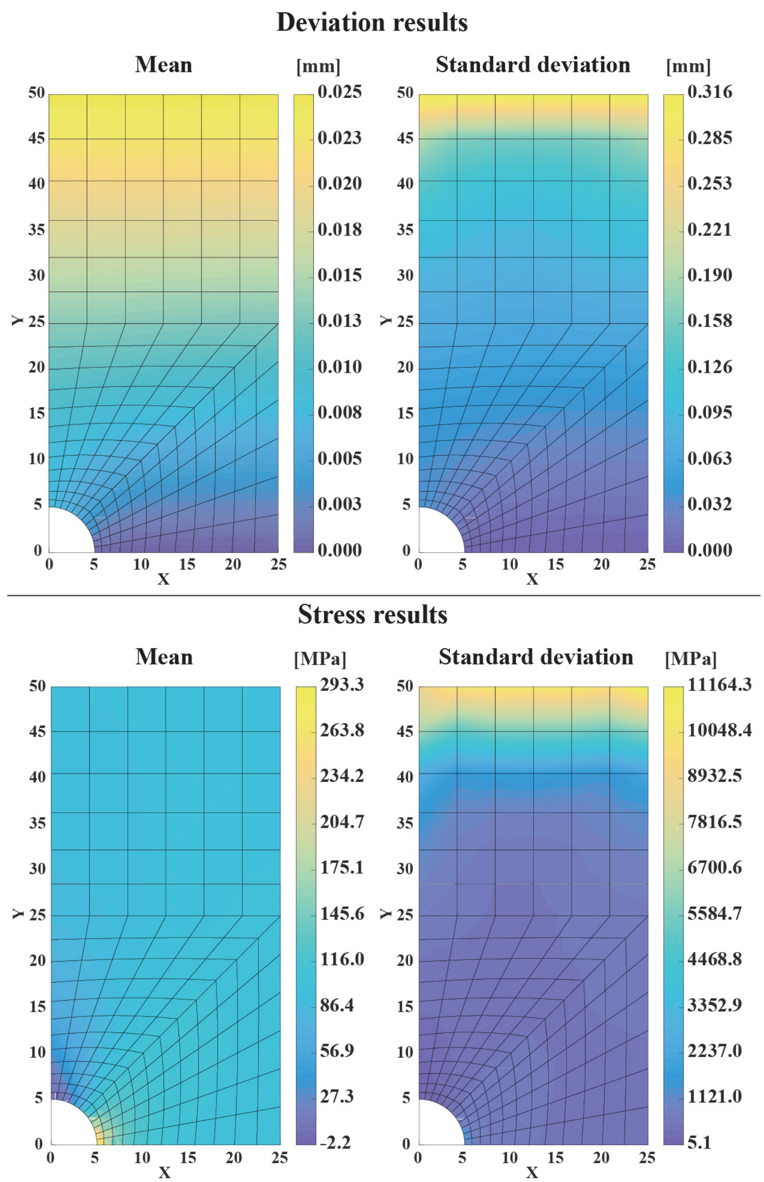

Fig. 3. Results of the first case study

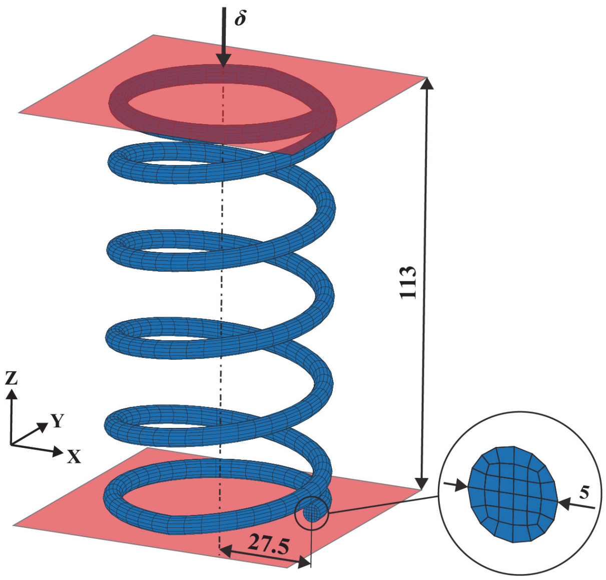

Fig. 4. Details of the second case study (dimensions in mm)

Only the top surface was set as mobile surface in order to compress the spring (see Fig. 4). The movement of the mobile surface is controlled by a single node placed in the centre of gravity of the surface. A deviation (δ) was considered on that node; the mean deviation vector $\{\mu\}_p$ foresees a deviation of -10 mm and the variance vector $\{\sigma^2\}_p$ foresees a variance of 1 mm^2 along the Z-direction. If the deviation is on a single node, the vectors $\{\mu\}_p$, $\{\sigma^2\}_p$ and the matrix $[K]_p$ have size (1×1), so they are scalars; for the same reason, the sensitivity matrices are (3m x 1) and (6m x 1) vectors for deviations and stresses respectively. Fig. 5 shows the mean deviation on the helical compression spring along Z-direction obtained by the unit force (a), the unit displacement (b) and super-element (c) methods to evaluate the sensitivity matrix $[S_\delta]$. Fig. 6 shows the von Mises stress on the helical compression spring obtained by the unit force (a), the unit displacement (b) and super-element (c) methods to evaluate the sensitivity matrix $[S_{\psi,\tau}]$. The mean deviations and stresses were evaluated by Eq.(10); the von Mises stress was evaluated by Eq.(14).

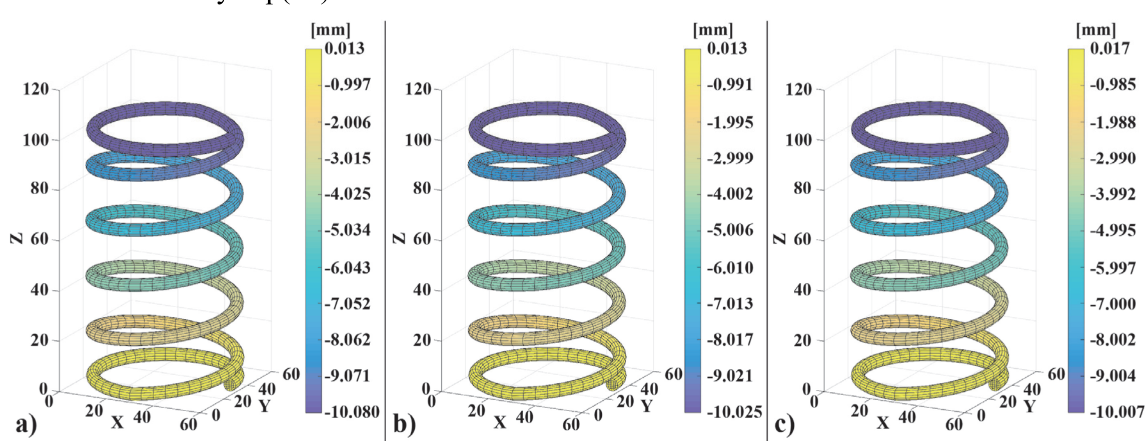

Fig. 5. Results of the second case study in terms of mean deviation by using: a) First method, b) Second method, c) Third method.

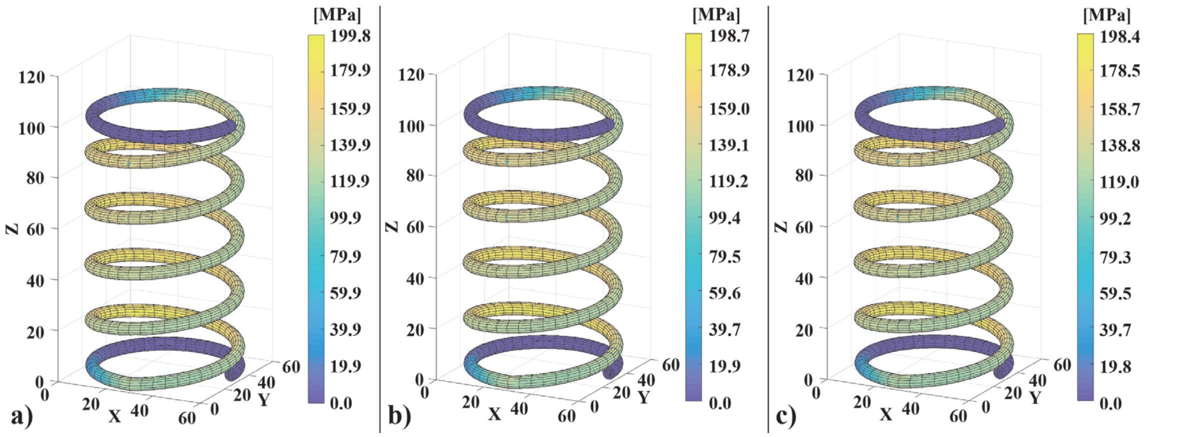

Fig. 6. Results of the second case study in terms of von Mises stress by using: a) First method, b) Second method, c) Third method.

A maximum error of 2% on both mean and standard deviation values for deviations and stresses was found between one method and another. Moreover, a reduction of simulation time was found comparing the different methods, as shown in Table 2. In this case, the method presented in §2.3 to evaluate $\left[K\right]_p$ allows to only reduce of about 18.6% the simulation time to perform the complete analysis. With the characteristics of the helical compression spring taken into account, the stiffness matrix is equal to 7.22 MPa according to Eq. (16). Similar values were also obtained numerically. In fact, the stiffness matrix $\left[K\right]_p$ obtained by the unit force, the unit displacement and super-element methods is equal to 7.37MPa, 7.33 MPa and 7.33 MPa respectively.

Table 2. Comparison between different methods in terms of simulation time for the second case study.

Time [s]				**Time reduction [%]**	
	Method			**§0 vs. §0**	**§0 vs. §0**
	§0	**§0**	**§0**		
$\left[K\right]_p$	18.7	21.6	10.7	15.5	-42.8
$\left[S_\delta\right],\left[S_{\psi,\tau}\right]$	41.9	45.7	34.2	9.1	-18.4
$\left\{\mu_\delta\right\}_a,\left\{\mu_{\psi,\tau}\right\}_a$ $\left\{\sigma_\delta^2\right\}_a,\left\{\sigma_{\psi,\tau}^2\right\}_a$	42.0	45.8	34.2	9.0	-18.6

4. Further remarks

The three methods, presented and described in Section 2, are based on the same hypotheses, i.e. small displacements and the linear-elastic behaviour of the materials. On these hypotheses the relationship between the assembly deviations and part deviations described in Eq. (1) is based. For this reason, the three methods have the same characteristics, so the same advantages and disadvantages.The evaluation of $\left[K\right]_p$ is the only aspect that differentiates them. In particular, the methods described in §2.2 and §2.3 evaluates $\left[K\right]_p$ directly with more or less FEA simulations respectively; the method described in §2.1 foresee also the inversion of matrix $\left[C\right]$ to obtain $\left[K\right]_p$. The main characteristics of the three methods were summarized in Table 3. It is evident that, if we consider that the time of a simulation is more or less the same, the method described in §2.3 requires less simulation time to evaluate $[K]_p$ for the same case study, i.e. (1 + *n*) times the time of a simulation is lower than (*n* + *n*) times the time of a simulation."

The great advantage to use these three methods is to evaluate, thanks to the sensitivity matrices, the deviations and stresses distribution in a time greatly reduced if compared to that required by DMC method. By knowing the stress distribution through the three compared methods, it is possible to assure that the stresses inside the product are below the yield strength by avoiding permanent deformation and the following material degradation. However, due to the previous hypotheses, it is clear that plastic strains are a source of deviation from nominal that are not captured in these simulation methods as well as the brittle damage which may occur before yielding. The sensitivity matrices are generally evaluated in the nominal condition, so they are constant. However, they may not be constant in highly nonlinear products, where plastic strains or brittle damage occur, for example. The DMC method is more flexible in solving nonlinear problems, but computationally very expensive.

Table 3. Number of FEA simulations for the three methods

To calculate	Method		
	§2.1	§2.2	§2.3
$[K]_p$	n	n	1
$[S_\delta],[S_{\psi,\tau}]$	n	n	n
$\{\mu_\delta\}_a,\{\mu_{\psi,\tau}\}_a,\{\sigma_\delta^2\}_a,\{\sigma_{\psi,\tau}^2\}_a$	$n + n$	$n + n$	$1 + n$

As stated in the Introduction section, many efforts were made to manage the non-linearity by using MIC-based methods; however, nowadays, this is still a topic of study. Moreover, future works will include the introduction of residual stresses, due to the manufacturing processes, in the tolerance analysis by always focusing on the increase of the computational efficiency by reducing the simulation times. In fact, even if the MIC-based methods are faster than DMC method, solving the sensitivity matrices is significant time consuming. In this perspective, another possible solution to define the sensitivity matrix $[S_\delta]$ in less time is to calculate it indirectly as $[S_\delta]=[K]_a^{-1}\cdot[K]_p$ by evaluating $[K]_a^{-1}$ and $[K]_p$ with the super-elements method.

5. Conclusions

The most used approach to solve tolerance analysis problems for flexible products is the method of influence coefficients that combines the finite element analysis with statistical analysis in order to establish a relationship between the product deviation and part deviation. The key of this relationship is the sensitivity matrix that can be evaluated through the matrix of influence coefficients that can be determined by the unit force method, the unit deviation method or the super-elements method.

A review of these methods was presented by showing how to evaluate the sensitivity matrix for stresses in order to also foresee the statistical distribution of stresses on products. Two single parts were used as case studies to review the methods by confirming the possibility to use these methods not only for tolerance analysis problems on flexible products but also to evaluate the statistical distribution of deviations and stresses due to operating conditions on single parts. Tensile and compressive conditions were considered as operating conditions and they were simulated by imposed and variable displacements. The obtained results show negligible variations between methods in the evaluation of statistical distribution of deviations and stresses and a better simulation time when the super-elements method is used.

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