Engineering Solid Mechanics 2 (2013) 29-42

Contents lists available at GrowingScience

Engineering Solid Mechanics

homepage: www.GrowingScience.com/esm

Free vibration of simply supported rectangular plates on Pasternak foundation: An exact and three-dimensional solution

M. Dehghany* **and A. Farajpour**

Young Researchers and Elites Club, South Tehran Branch, Islamic Azad University, Tehran, Iran

A R T I C L E I N F O

Article history:
Received September 20, 2013
Received in Revised form October, 14, 2013
Accepted 15 December 2013
Available online 20 December 2013

Keywords:
Simply supported rectangular plates
Elastic foundation
Pasternak model
Three-dimensional elasticity theory
Free vibration
Eigen-frequency

A B S T R A C T

This paper deals with exact solution for free vibration analysis of simply supported rectangular plates on elastic foundation. The solution is on the basis of three dimensional elasticity theory. The foundation is described by the Pasternak (two-parameter) model. First, the Navier equations of motion are replaced by three decoupled equations in terms of displacement components. Then, these equations are solved in a semi-inverse method. The obtained displacement field satisfies all the boundary conditions of the problem in a point wise manner. The solution is in the form of a double Fourier sine series. Then free-vibration characteristics of rectangular plates resting on elastic foundations for different thickness/span ratios and foundation parameters are studied. The numerical results are compared with the available results in the literature. Important parameters on the accuracy of plate theories and free-vibration characteristics of rectangular plates resting on elastic foundations are discussed.

1. Introduction

Rectangular plates resting on elastic foundation are frequently used structural elements in modeling of engineering problems such as: concrete roads, mat foundations of buildings and reinforced concrete pavements of airport runways. The Pasternak model (Pasternak, 1954) or the two-parameter model is frequently adopted to describe the mechanical behavior of foundations. The well known Winkler model (Winkler, 1867) can be considered as a special case that ignores the shear deformation of the foundation. Depending on the plate thickness, two main theories may be considered for modeling a rectangular plate. Classical and first-order shear deformation plate theories for thin and moderately thick rectangular plates, respectively. The classical plate theory, referred to as

* Corresponding author. Tel.: +98 21 7724 0201
E-mail addresses: dehghany.Dahaj65@gmail.com (M. Dehghani)

doi: 10.5267/j.esm.2013.12.001

Kirchhoff's theory (Timoshenko & Woinowsky-Krieger, 1970), does not take into account either the effect of transverse shear deformation or rotatory inertia, and hence it becomes inaccurate for thicker plates. The first-order shear deformation theory was proposed by Reissner (1945), and developed further for the deformable plates in statics and dynamics by Mindlin (1951), Mindlin and Deresiewicz (1954) and Mindlin et al. (1956). They considered both these effects, by assuming linear variations for all three displacement components across the thickness. Although first-order shear deformation theory is more accurate than classical plate theory, this theory cannot provide accurate results for thick plates. The inherent deficiency of the Mindlin plate theory is the presence of a correction coefficient k^2, which is introduced to correct the contradictory shear stress distribution over the thickness of the plate that cannot be found from the assumptions of the theory itself. Moreover, in order to improve the solution accuracy, some higher-order plate theories were also formulated (Hanna & Leissa, 1994; Reddy, 1984). However, two dimensional plate theories cannot be exact because these theories neglect the transverse normal and shear stress effects.

Nomenclature

a, b	Length and width of plate
D	Flexural rigidity
e	Volume expansion coefficient
E	Young's modulus
h	Plate thickness
i	$\sqrt{-1}$
k_s	Shear foundation coefficient
k_t	Total foundation coefficient
k_w	Winkler foundation coefficient
m, n	The wave numbers in the x and y directions, respectively
t	Time
u, v, w	Displacements in the x, y and z directions, respectively
x, y, z	Cartesian coordinate system
∇	Gradient operator (dell)
$\Delta_{m,n}$	Eigenfrequency parameter
λ, μ	Lame's elastic constants
ϕ_1, ϕ_2	Non-dimensional foundation parameters
ν	Poisson's ratio
ρ	Mass density
σ_x, σ_y, σ_z	Normal stresses
τ_{xy}, τ_{xz}, τ_{yz}	Shear stresses
ϖ	Natural radian frequency

Two-dimensional approximate plate theories were widely used to investigate the vibrational behavior of rectangular plates rested on the Pasternak foundation. For example, Lam et al. (2000) investigated the elastic bending, buckling and vibration of Levy-plates resting on a two-parameter foundation using classical plate theory and Green's functions. Xiang et al. (1994) studied the vibration problem of initially stressed thick rectangular plates on Pasternak foundation using Mindlin plate theory. Omurtag et al. (1997) explored the free vibration of thin rectangular plates resting on Pasternak elastic foundation with variable thickness and different boundary and loading conditions on the basis of classical plate theory. Wen (2007) investigated the method of fundamental solution applied to the rectangular Mindlin plates resting on the Pasternak foundation. Zhong and Yin (2008) explored eigen-frequencies and vibration modes of a rectangular thin plate on an elastic foundation with completely free boundary using classical plate theory and integral transform method. Matsunaga (2000) studied the vibration and stability of simply supported rectangular thick plates on a Pasternak foundation using a special higher-order plate theory. Akhavan et al. (2009) investigated vibrational behavior of rectangular plates resting on Pasternak foundation on the basis of Mindlin plate theory. Hosseini Hashemi et al. (2010) investigated the free vibration analysis of vertical rectangular Mindlin plates resting on Pasternak elastic foundation and fully or partially in contact with fluid on their one side. This analysis has been done for different combinations of boundary conditions.

As mentioned earlier, two dimensional theories are inherently erroneous. In order to overcome this disadvantage, several attempts have been made for three-dimensional vibration analysis of rectangular thick plates. For example, Lim et al. (1998 a,b) investigated the numerical aspects for free vibration analysis of thick plates. In their research works, the effect of transverse shear strain has been considered by using a higher-order plate theory without the need for a shear correction factor. Levinson (1985) presented an exact, three-dimensional solution for the free vibrations of simply supported, rectangular plates of arbitrary thickness within the linear theory of elasto-dynamic. Srinivas and Rao (1970) presented an exact closed form characteristic equation for obtaining natural frequencies of thick, simply supported homogeneous or laminated rectangular plates. Furthermore, recently the vibration analysis of a functionally graded rectangular plate resting on two parameter elastic foundation has been done by Hasani Baferani et al. (2011) employing the third order shear deformation plate theory. Free vibration of exponentially graded sandwich plates resting on elastic foundations was the subject of another research by Sobhy (2012).

Despite extensive two-dimensional approximate studies on the vibration of rectangular plates on elastic foundation, very few researches can be found for exact three-dimensional vibration analysis of plates on elastic foundation. For instance, Zhou et al. (2004) studied the vibration of rectangular thick plates on Pasternak foundation based on the Ritz method. In addition, Tajeddini et al. (2011) investigated free vibration of thick circular and annular plates resting on Pasternak foundation. Malekzadeh (2009) also adopted differential quadrature method (DQM) and series solution to study free vibration of thick functionally graded plates supported on two-parameter elastic foundation. Furthermore, a global transfer matrix and Durbin's numerical Laplace inversion algorithm were employed by Hasheminejad and Gheshlaghi (2012) to study the transient vibration of simply supported, functionally graded rectangular plates resting on a linear Winkler-Pasternak viscoelastic foundation. In this paper, an exact three dimensional linear elasticity solution for free vibration analysis of simply supported rectangular plates resting on Pasternak foundation is presented.

2. Formulation

It is well known that for linear elasticity, a problem may be reduced to Navier equations of motion together with appropriate boundary and initial conditions. When body forces are absent these equations in the *x, y* and *z* directions, respectively, are:

$$\mu\nabla^2 u+(\lambda+\mu)\frac{\partial e}{\partial x}=\rho\ddot{u} \tag{1a}$$

$$\mu\nabla^2 v+(\lambda+\mu)\frac{\partial e}{\partial y}=\rho\ddot{v} \tag{1b}$$

$$\mu\nabla^2 w+(\lambda+\mu)\frac{\partial e}{\partial z}=\rho\ddot{w} \tag{1c}$$

Here λ and μ are the Lame's elastic constants, which are related to the Young's modulus E and Poisson ratio ν by

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)};\ \mu=\frac{E}{2(1+\nu)} \tag{2}$$

ρ is the mass density of the elastic body, *u, v* and *w* are the components of the displacement vector in the *x, y* and *z* directions, respectively. The "dot" above each parameter denotes differentiation with respect to time. ∇ is the gradient operator usually called "del" and *e* (volume expansion coefficient) is

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \varepsilon_x + \varepsilon_y + \varepsilon_z \, , \tag{3}$$

which is invariant with respect to the rotation of the coordinate axes. From Eq. (1), it is obvious that this parameter (*e*) couples the displacement components in each of the Navier equations. Apparently, using decoupled forms of the governing equations of each theory for solving related problems is much easier than using the original coupled form for both analytical and numerical methods. Thus, in the next section first the Navier equations of motion are decoupled in terms of displacement components.

2.1. Decoupling of Navier equations

In order to decouple the Navier equations, the following procedure is proposed:
Differentiating Eqs. (1a-c) with respect to *x, y* and *z* respectively, and adding the results yields:

$$\mu\nabla^2 e + (\lambda + \mu)\nabla^2 e = \rho\ddot{e} \Rightarrow ((\lambda + 2\mu)\nabla^2 - \rho\frac{\partial^2}{\partial t^2})e = V[e] = 0 \tag{4}$$

in which, $(\lambda+2\mu)\nabla^2 - \rho\frac{\partial^2}{\partial t^2}$ is named as V operator, which can vanish volume expansion coefficient and hence is able to decouple the Navier equations of motion. In fact, Eq. (4) confirms that the volume expansion *e* can be interpreted as a wave that propagates in an elastic medium with a constant speed of $((\lambda+2\mu)/\rho)^{0.5}$ (Timoshenko & Goodier, 1951). Imposing V operator into Eqs. (1a), (1b) and (1c) yields:

$$\mu(\lambda + 2\mu)\nabla^4 u - \rho(\lambda + 3\mu)\nabla^2 \ddot{u} + \rho^2 \frac{\partial^4 u}{\partial t^4} = 0 \tag{5a}$$

$$\mu(\lambda + 2\mu)\nabla^4 v - \rho(\lambda + 3\mu)\nabla^2 \ddot{v} + \rho^2 \frac{\partial^4 v}{\partial t^4} = 0 \tag{5b}$$

$$\mu(\lambda + 2\mu)\nabla^4 w - \rho(\lambda + 3\mu)\nabla^2 \ddot{w} + \rho^2 \frac{\partial^4 W}{\partial t^4} = 0 \tag{5c}$$

These equations were also obtained by Saidi et al. (2009) via another complicated method.

2.2. Semi-inverse method

In order to analyze free vibration of simply supported rectangular plates on Pasternak foundation the following assumptions for displacement field were used.

$$w(x,y,z,t) = W_{mn} f(z)\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i\varpi t} \tag{6a}$$

$$u(x,y,z,t) = -g(z)\frac{\partial w}{\partial x} \tag{6b}$$

$$v(x,y,z,t) = -g(z)\frac{\partial w}{\partial y} \tag{6c}$$

where *f(z)* and *g(z)* are two unknown functions of *z* and *a,b* are the length and width of the plate respectively. In order to satisfy the transverse displacement boundary conditions on the edges of a simply supported plate, *m* and *n* are taken to be integers. ϖ is the natural radian frequency and $i=\sqrt{-1}$.

Substituting Eqs. (6a-c) into Eqs. (5a-c) and assuming that $f\ (z)$ and $g\ (z)$ are proportional to $e^{\beta z}$, where β is a constant, leads to the following equation.

$$\beta^4 - [2(M\pi)^2 - \frac{(\lambda+3\mu)}{\mu(\lambda+2\mu)}\rho\varpi^2]\beta^2 + [(M\pi)^4 - \frac{(\lambda+3\mu)}{\mu(\lambda+2\mu)}(M\pi)^2\rho\varpi^2 + \frac{\rho^2\varpi^4}{\mu(\lambda+2\mu)}] = 0, \tag{7}$$

where the parameter M is defined as:

$$M^2 = (\frac{m}{a})^2 + (\frac{n}{b})^2 \tag{8}$$

Eq. (7) is a characteristic equation for the elastodynamic problem of the simply supported rectangular plate (Levinson 1985), which describes the mechanical behavior of these components for arbitrary values of ϖ. The roots of Eq. (8) are the relatively simple expressions:

$$\beta_1^2 = (M\pi)^2 - \frac{\rho\varpi^2}{\mu} \tag{9a}$$

$$\beta_2^2 = (M\pi)^2 - \frac{\rho\varpi^2}{(2\mu+\lambda)} \tag{9b}$$

Depending on ϖ^2 values, the following five different possible cases arise.

$$\text{Case (i)} \quad \varpi^2 < \frac{(M\pi)^2\mu}{\rho} \quad (\beta_1^2, \beta_2^2 > 0) \tag{10a}$$

$$\text{Case (ii)} \quad \varpi^2 = \frac{(M\pi)^2\mu}{\rho} \quad (\beta_1 = 0, \beta_2^2 > 0) \tag{10b}$$

$$\text{Case (iii)} \quad \frac{(M\pi)^2\mu}{\rho} < \varpi^2 < \frac{(M\pi)^2(\lambda+2\mu)}{\rho} \quad (\beta_1^2 < 0, \beta_2^2 > 0) \tag{10c}$$

$$\text{Case (iv)} \quad \varpi^2 = \frac{(M\pi)^2(\lambda+2\mu)}{\rho} \quad (\beta_1^2 < 0, \beta_2 = 0) \tag{10d}$$

$$\text{Case (v)} \quad \frac{(M\pi)^2(\lambda+2\mu)}{\rho} < \varpi^2 \quad (\beta_1^2 < 0, \beta_2^2 < 0) \tag{10e}$$

Considering the above five cases one can obtain $f(z)$ and $g(z)$ functions as below:

$$\text{Case (i)} \quad \begin{aligned} f(z) &= k_1 + k_2 z + k_3 \cosh\beta_2 z + k_4 \sinh\beta_2 z \\ g(z) &= l_1 + l_2 z + l_3 \cosh\beta_2 z + l_4 \sinh\beta_2 z \end{aligned} \tag{11a}$$

$$\text{Case (ii)} \quad \begin{aligned} f(z) &= k_1 + k_2 z + k_3 \cosh\beta_2 z + k_4 \sinh\beta_2 z \\ g(z) &= l_1 + l_2 z + l_3 \cosh\beta_2 z + l_4 \sinh\beta_2 z \end{aligned} \tag{11b}$$

$$\text{Case (iii)} \quad \begin{aligned} f(z) &= k_1 \cos\beta_1 z + k_2 \sin\beta_1 z + k_3 \cosh\beta_2 z + k_4 \sinh\beta_2 z \\ g(z) &= l_1 \cos\beta_1 z + l_2 \sin\beta_1 z + l_3 \cosh\beta_2 z + l_4 \sinh\beta_2 z \end{aligned} \tag{11c}$$

$$\text{Case (iv)} \quad \begin{aligned} f(z) &= k_1 \cos\beta_1 z + k_2 \sin\beta_1 z + k_3 + k_4 z \\ g(z) &= l_1 \cos\beta_1 z + l_2 \sin\beta_1 z + l_3 + l_4 z \end{aligned} \tag{11d}$$

$$\text{Case (v)} \quad \begin{aligned} f(z) &= k_1 \cos\beta_1 z + k_2 \sin\beta_1 z + k_3 \cos\beta_2 z + k_4 \sin\beta_2 z \\ g(z) &= l_1 \cos\beta_1 z + l_2 \sin\beta_1 z + l_3 \cos\beta_2 z + l_4 \sin\beta_2 z \end{aligned} \tag{11e}$$

Using displacement field, which is mentioned before, the strain-displacement relation and Hooke's law for a linear elastic isotropic material the stress field can be found as:

$$
\begin{aligned}
\sigma_x &= W_{mn}[(\lambda M^2\pi^2 + 2\mu(\frac{m\pi}{a})^2)g + \lambda f']\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i\varpi t}, \\
\sigma_y &= W_{mn}[(\lambda M^2\pi^2 + 2\mu(\frac{n\pi}{b})^2)g + \lambda f']\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i\varpi t}, \\
\sigma_z &= W_{mn}[(\lambda M^2\pi^2)g + (\lambda+2\mu)f']\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i\varpi t}, \\
\tau_{xy} &= -2\mu W_{mn}(\frac{mn\pi^2}{ab})g\cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{i\varpi t}, \\
\tau_{xz} &= \mu W_{mn}(\frac{m\pi}{a})(f-g')\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})e^{i\varpi t}, \\
\tau_{yz} &= \mu W_{mn}(\frac{n\pi}{b})(f-g')\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})e^{i\varpi t}.
\end{aligned}
\tag{12}
$$

Using semi inverse method by displacement field as an initial assumption, we should satisfy boundary conditions and equilibrium equations to obtain an exact three dimensional elasticity solution. First, the equilibrium equations are considered as below:

$$
\begin{aligned}
&\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = \rho\ddot{u} \\
&\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z} = \rho\ddot{v} \\
&\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} = \rho\ddot{w}
\end{aligned}
\tag{13}
$$

Substitution of the assumed stress field (Eq. (12)) into the equilibrium equations leads to the following equations for *f (z)* and *g (z)*

$$\mu g'' - (\lambda+\mu)f' - (\lambda+2\mu)[(M\pi)^2 - \frac{\rho\varpi^2}{(\lambda+2\mu)}]g = 0, \tag{14a}$$

$$(\lambda+2\mu)f'' + (\lambda+\mu)(M\pi)^2 g' - \mu[(M\pi)^2 - \frac{\rho\varpi^2}{\mu}]f = 0. \tag{14b}$$

Eq. (14a) is obtained from the equilibrium equations in both *x* and *y* directions while Eq. (14b) comes from the equilibrium equation in the *z* direction. Substituting Eqs. (11a-e) into Eqs. (14a) and (14b) will show that the arbitrary constants k_j, l_j (j=1, 2, 3, 4) are not independent and related to each other as below:

$$\text{Case (i)} \quad l_1 = \frac{-\beta_1 k_2}{(M\pi)^2}, l_2 = \frac{-\beta_1 k_1}{(M\pi)^2}, l_3 = \frac{-k_4}{\beta_2}, l_4 = \frac{-k_3}{\beta_2}, \tag{15a}$$

$$\text{Case (ii)} \quad l_1 = -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{k_2}{\beta_2^2}, l_2 = 0, l_3 = \frac{-k_4}{\beta_2}, l_4 = \frac{-k_3}{\beta_2}, \tag{15b}$$

$$\text{Case (iii)} \quad l_1 = \frac{-\beta_1 k_2}{(M\pi)^2}, l_2 = \frac{\beta_1 k_1}{(M\pi)^2}, l_3 = \frac{-k_4}{\beta_2}, l_4 = \frac{-k_3}{\beta_2}, \tag{15c}$$

$$\text{Case (iv)} \quad l_1 = \frac{-\beta_1 k_2}{(M\pi)^2}, l_2 = \frac{\beta_1 k_1}{(M\pi)^2}, k_4 = 0, l_4 = \frac{-k_3\mu\beta_1^2}{(M\pi)^2(\lambda+\mu)}, \tag{15d}$$

$$\text{Case (v)} \quad l_1 = \frac{-\beta_1 k_2}{(M\pi)^2}, l_2 = \frac{\beta_1 k_1}{(M\pi)^2}, l_3 = \frac{k_4}{\beta_2}, l_4 = \frac{-k_3}{\beta_2}, \tag{15e}$$

in which β_1, β_2 has been written for $|\beta_1|$, $|\beta_2|$, Eqs. (15a-e) show that in each case, only four constants are independent. These constants must be determined using boundary conditions. The plate geometry and appropriate boundary conditions are elaborated in the next section.

2.3. Boundary conditions

Consider a homogeneous isotropic rectangular thick plate with length a, width b and thickness h, which is resting on an elastic foundation as shown in Fig. 1. A Cartesian coordinate system is used to describe the plate geometry and dimensions such that the origin is at the mid-plate corner and the axes *(x, y, z)* are parallel to the edges of the plate. Furthermore, the Pasternak model is used to describe the reaction of the foundation on the plate and hence boundary conditions are:

$$\tau_{zx} = \tau_{zy} = 0 \text{ at } z = \frac{h}{2}, -\frac{h}{2}$$

$$\sigma_z = 0 \text{ at } z = \frac{-h}{2}$$

$$\sigma_z = -k_w w + k_s \nabla_2^2 w \text{ at } z = \frac{h}{2} \tag{16}$$

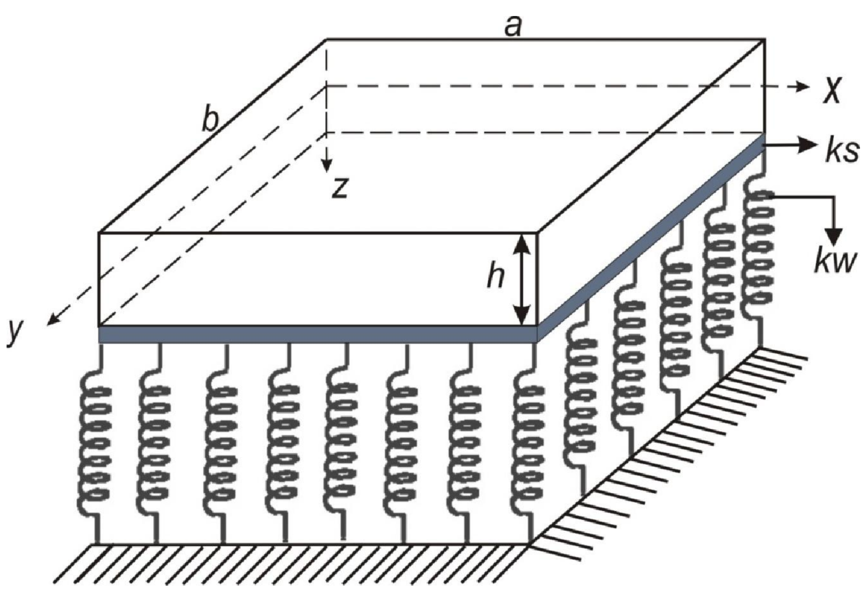

Fig 1. A rectangular plate on a two-parameter elastic foundation

Imposing the boundary conditions in Eq. (12) yields:

$$f - g' = 0 \quad \text{at} \quad z = \frac{-h}{2}, \frac{h}{2}$$

$$(\lambda M^2 \pi^2) g + (\lambda + 2\mu) f' = 0 \quad \text{at} \quad z = \frac{-h}{2}$$

$$(\lambda M^2 \pi^2) g + (\lambda + 2\mu) f' = -k_t f(z) \quad \text{at} \quad z = \frac{h}{2} \tag{17}$$

in which k_t is a parameter that can be calculated from the following equation:

$$\sigma_z = -k_w w + k_s \nabla_2^2 w = (-k_w - (M\pi)^2 k_s) w = -k_t w \Rightarrow k_t = (k_w + (M\pi)^2 k_s) \tag{18}$$

in which k_w is the Winkler foundation stiffness and k_s is the shear stiffness of the elastic foundation. It is obvious that this parameter (k_t) depends on the plate dimensions and the mode shapes of the vibration in addition to Winkler and shear foundation coefficients. One can interpret this parameter as a total foundation coefficient for simply supported rectangular plates resting on a Pasternak foundation.

2.4. Determination of natural radian frequencies

Satisfaction of the boundary conditions of relation (17) yields a set of four homogeneous equations for each (*m, n*). These can be put in the form

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

in which the square matrix $[q_{\alpha\beta}]$, α, β=1, 2, 3 and 4 for three important cases i.e. cases (i), (iii), (v) (Levinson 1985) are given in the appendix. For a non-trivial solution of the problem (Eq. (19)), the determinant of the square matrix $[q_{\alpha\beta}]$ must be zero and this yields an eigenvalue problem in each case. Solving the obtained eigenvalue problem for selected values of *m, n* gives the corresponding desirable natural radian frequencies (ϖ). It is necessary to note that the obtained modes of vibration are those consistent with the kinematic assumptions of Eq. (6). It means that it should not be assumed that no other modes of free oscillation for such plates exist within the three dimensional theory of linear elasto-dynamics.

3. Numerical examples

For the purposes of illustrating the performed analysis in the previous section and making comparisons with the predictions of the other theories, the results for square plates with simply supported edges on Pasternak foundation are presented in this section. For the sake of comparison with data presented by Zhou et al. (2004), non-dimensional eigen-frequency parameter Δ and non-dimensional foundation parameters ϕ_1 and ϕ_2 are defined as bellow:

$$\Delta = \frac{\varpi b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}, \quad \phi_1 = \frac{k_w a^4}{D}, \quad \phi_2 = \frac{k_s a^2}{D}, \tag{20}$$

where $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate. Moreover, the eigen-frequency parameter $\Delta_{m,n}$ $(m, n=1, 2, 3 \ldots)$ as defined in Eq. (20) is used to denote the flexural modes of the plate where subscripts *m* and *n* mean the wave numbers in the *x* and *y* directions, respectively (Zhou et al., 2004).

Using proposed method, eigen-frequency parameters for simply supported thin (*h/b=0.01*) and moderately thick (*h/b=0.1*) square plates on Pasternak foundation have been calculated. The obtained results, the results of the Ritz method (Zhou et al. 2004) and the exact Mindlin plate solutions (Xiang et al. 1994) are given in Table 1. This Table shows that although for thin plates (*h/b=0.01*) on Pasternak foundation, both Ritz method and Mindlin plate theory can give accurate results, the Ritz method provide more accurate results for moderately thick plates (*h/b=0.1*).

Tables 2 and 3 show the obtained results for simply supported thick square plates on Winkler and Pasternak foundations, respectively. In these Tables, the obtained results for a moderately thick plate (*h/b=0.2*) are compared with the results of Ritz method (Zhou et al. 2004), Mindlin plate theory

(Xiang et al. 1994) and the special higher-order plate theory (Matsunaga 2000). Moreover, the obtained results for a thick plate with thickness/width ratio (*h/b=0.5*) are also compared with the result of Ritz method (Zhou et al. 2004) and the special higher-order plate theory (Matsunaga 2000).

Table 1. Comparison between the obtained results and the results of the other theories for thin and moderately thick simply supported square plates on Pasternak foundation (ϕ_2=10, case (i))

h/b	ϕ_1	Method	$\Delta_{1,1}$	$\Delta_{1,2}$,$\Delta_{2,1}$	$\Delta_{1,3}$,$\Delta_{3,1}$
.01	100	Exact 3D	2.6550	5.5717	8.5406
		Ritz method *	2.6551	5.5717	8.5406
		Mindlin **	2.6551	5.5718	8.5405
.01	500	Exact 3D	3.3398	5.9285	8.7775
		Ritz method	3.3398	5.9285	8.7775
		Mindlin	3.3400	5.9287	8.7775
0.1	200	Exact 3D	2.7756	5.2953	7.7272
		Ritz method	2.7756	5.2954	7.7279
		Mindlin	2.7842	5.3043	7.7287
0.1	1000	Exact 3D	3.9566	5.9756	8.1947
		Ritz method	3.9566	5.9757	8.1954
		Mindlin	3.9805	6.0078	8.2214

* Zhou et al. (2004)
** Xiang et al. (1994)

Table 2. Comparison between the obtained results and the results of the other theories for moderately thick and thick simply supported square plates on Winkler foundation (ϕ_2=0, case (i))

h/b	ϕ_1	Method	$\Delta_{1,1}$	$\Delta_{1,2}$,$\Delta_{2,1}$	$\Delta_{1,3}$,$\Delta_{3,1}$
0.2	10	Exact 3D	1.8020	3.9103	6.6930
		Ritz method *	1.8020	3.9103	6.6930
		Mindlin **	1.7955	3.8780	6.6078
		Higher-order ***	1.8020	3.9103	6.6930
0.2	10^2	Exact 3D	2.0216	4.0090	6.7479
		Ritz method	2.0216	4.0090	6.7479
		Mindlin	2.0268	3.9875	6.6719
		Higher-order	2.0216	4.0090	6.7479
0.2	10^3	Exact 3D	3.4793	4.8499	7.2503
		Ritz method	3.4793	4.8499	7.2503
		Mindlin	3.5972	4.9499	7.2812
		Higher-order	3.4793	4.8499	7.2503
0.2	10^4	Exact 3D	4.6127	7.2934	9.8784
		Ritz method	4.6127	7.2934	9.8785
		Mindlin	9.9835	10.430	11.689
		Higher-order	4.6127	7.2934	9.8785
0.2	10^5	Exact 3D	4.6127	7.2934	10.314
		Ritz method	4.6127	7.2934	10.314
		Mindlin	17.990	20.089	23.075
		Higher-order	4.6127	7.2934	10.314
0.5	10	Exact 3D	1.2903	2.3479	3.5549
		Ritz method	1.2903	2.3480	3.5550
		Higher-order	1.2903	2.3480	3.5550
0.5	10^2	Exact 3D	1.5026	2.4674	3.6376
		Ritz method	1.5026	2.4674	3.6376
		Higher-order	1.5026	2.4674	3.6376
0.5	10^3	Exact 3D	1.8451	2.7689	3.8276
		Ritz method	1.8451	2.7689	3.8276
		Higher-order	1.8451	2.7689	3.8276
0.5	10^4	Exact 3D	1.8451	2.8733	3.8860
		Ritz method	1.8451	2.8733	3.8860
		Higher-order	1.8451	2.8733	3.8860
0.5	10^5	Exact 3D	1.8451	2.9174	3.8927
		Ritz method	1.8451	2.8857	3.8927
		Higher-order	1.8451	2.8857	3.8927

* Zhou et al. (2004)
** Xiang et al. (1994)
*** Matsunaga (2000)

Table 3. Comparison between the obtained results and the results of the other theories for thick simply supported square plates on Pasternak foundation (ϕ_2=10, case (i))

h/b	ϕ_1	Method	$\Delta_{1,1}$	$\Delta_{1,2}$,$\Delta_{2,1}$	$\Delta_{1,3}$,$\Delta_{3,1}$
0.2	0	Exact 3D	2.2334	4.4056	7.2436
		Ritz method *	2.2334	4.4056	7.2436
		Mindlin **	2.2505	4.4344	7.2727
		Higher-order ***	2.2334	4.4056	7.2436
0.2	10	Exact 3D	2.2539	4.4150	7.2487
		Ritz method	2.2539	4.4150	7.2487
		Mindlin	2.2722	4.4452	7.2792
		Higher-order	2.2539	4.4150	7.2488
0.2	10^2	Exact 3D	2.4300	4.4986	7.2948
		Ritz method	2.4300	4.4986	7.2948
		Mindlin	2.4591	4.5409	7.3373
		Higher-order	2.4300	4.4986	7.2948
0.2	10^3	Exact 3D	3.7111	5.2285	7.7191
		Ritz method	3.7111	5.2285	7.7191
		Mindlin	3.8567	5.4043	7.8938
		Higher-order	3.7112	5.2285	7.7191
0.2	10^4	Exact 3D	4.6127	7.2934	10.033
		Ritz method	4.6127	7.2934	10.033
		Mindlin	10.076	10.644	12.067
		Higher-order	4.6127	7.2934	10.033
0.5	0	Exact 3D	1.6462	2.6851	3.8268
		Ritz method	1.6462	2.6851	3.8268
		Higher-order	1.6462	2.6851	3.8268
0.5	10	Exact 3D	1.6577	2.6879	3.8274
		Ritz method	1.6577	2.6879	3.8274
		Higher-order	1.6577	2.6879	3.8274
0.5	10^2	Exact 3D	1.7437	2.7096	3.8321
		Ritz method	1.7437	2.7096	3.8321
		Higher-order	1.7437	2.7096	3.8321
0.5	10^3	Exact 3D	1.8451	2.8033	3.8578
		Ritz method	1.8451	2.8033	3.8578
		Higher-order	1.8451	2.8033	3.8578
0.5	10^4	Exact 3D	1.8451	2.8739	3.8866
		Ritz method	1.8451	2.8739	3.8866
		Higher-order	1.8451	2.8739	3.8866
0.5	10^5	Exact 3D	1.8451	2.8857	3.8927
		Ritz method	1.8451	2.8857	3.8927
		Higher-order	1.8451	2.8857	3.8927

* Zhou et al. (2004)
** Xiang et al. (1994)
*** Matsunaga (2000)

From Tables 2 and 3 it can be observed that although both of the Ritz method and special higher-order plate theory can provide accurate results, Mindlin plate theory is unable to predict eigen-frequencies accurately. It is obvious that the accuracy of Mindlin plate theory decreases when the thickness-width ratio or the magnitude of the foundation stiffness increases. This deviation is predictable since in the Mindlin plate theory, the foundation is assumed to be acting on the median surface of the plate. Therefore, increasing the plate thickness or foundation stiffness reduces the accuracy of the Mindlin plate theory. From Tables 1 to 3, it is obvious that the special higher-order plate theory and the Ritz method can provide very accurate results for vibration of simply supported rectangular thin, moderately thick and thick plates on elastic foundation. However, it is worth noting that these methods need solving a set of simultaneous partial differential equations or complicated numerical calculations, while the presented solution is exact, simple and straightforward. Therefore, it is more efficient to use the presented solution instead of the higher-order plate theory or the Ritz method.

Tables 4 and 5 show the eigen-frequencies for simply supported thick square plates (*h/b=0.6*) on Winkler foundation for case (iii), (v) respectively. It is obvious from Tables 1 to 5 that eigen-frequencies of the flexural modes generally increase with increasing the foundation stiffness. However, as the foundation stiffness is increased, the eigen-frequencies exhibit less variation and converge to a constant value. This constant value is independent of the foundation type (Winkler or Pasternak). This is because for higher foundation stiffness, the displacement w of the lower surface becomes nearly zero. For example, for case (i), these trends are presented in Fig. 2.

Table 4 Simply supported thick square plates on Winkler foundation (ϕ_2=0, case (iii))

h/b	ϕ_1	Method	$\Delta_{1,1}$
0.6	0	Exact	2.33161
		Mindlin *	2.37615
0.6	10^3	Exact	2.67481
0.6	10^4	Exact	2.72326
0.6	10^5	Exact	2.72836

* Levinson (1985)

Table 5 Simply supported thick square plates on Winkler foundation (ϕ_2=0, case (v))

h/b	ϕ_1	Method	$\Delta_{1,1}$
0.6	0	Exact	3.13308
0.6	10^3	Exact	3.24329
0.6	10^4	Exact	3.25076
0.6	10^5	Exact	3.25151

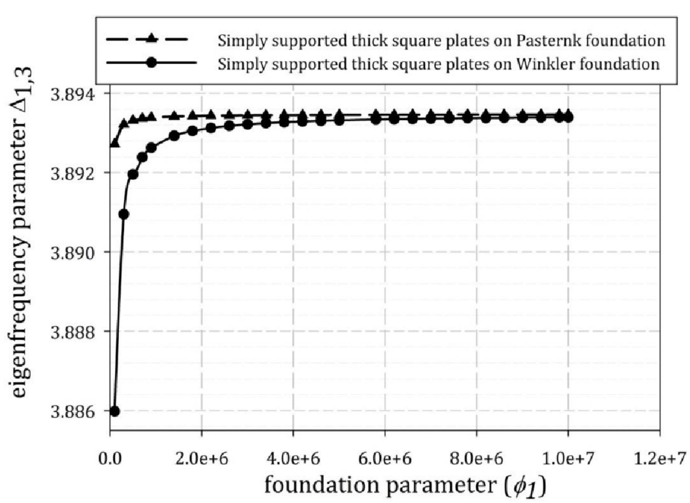

Fig 2. Variations of eigen-frequencies $\Delta_{1,3}$ in terms of foundation parameter (ϕ_1)(case (i))

4. Conclusions

The exact, linear elasto-dynamic analysis for certain modes (those are consistent with the kinematic assumptions of Eq. (6)) of vibration of a simply supported, rectangular plate resting on the Pasternak foundation has been presented. The conclusions are summarized as:

- The Novier equations of motion were decoupled using a new and simple method.
- These new decoupled equations were then employed in the semi inverse method by the displacement field presented in Eq. (6) as an initial assumption. It was shown that depending on the magnitude of the natural radian frequencies, three important different cases in mechanical behavior of the plate may be arising.
- In each case, satisfaction of the equilibrium equations and boundary conditions of the problem yielded an eigenvalue problem. After solving this problem, the natural radian frequencies and corresponding mode shapes of vibration can be obtained.
- Using the presented exact solution, natural radian frequencies for thin, moderately thick and thick square plates with simply supported edges on Pasternak foundation were calculated. Then, the obtained results were compared with the predictions of the other theories and the main trends were discussed.
- The presented solution is exact, simple and straightforward. Therefore, it is more efficient to use this method instead of the complicated numerical methods or approximated plate theories.

Appendix

In Eq. (19) $q_{\alpha\beta}$ (α, β=1, 2, 3, 4) are:

Case (i):

$$[q_{\alpha\beta}]=\begin{bmatrix} \frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_1 h}{2}) & 2\beta_1\beta_2\cosh(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_2 h}{2}) & p^2\cosh(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_2 h}{2}) \\ 2\beta_1\beta_2\sinh(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_1 h}{2}) & p^2\sinh(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_2 h}{2}) & \frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_2 h}{2}) \\ p^2\cosh(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\cosh(\frac{\beta_2 h}{2}) & 0 \\ 0 & p^2\sinh(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\sinh(\frac{\beta_2 h}{2}) \end{bmatrix}$$

Case (iii):

$$[q_{\alpha\beta}]=\begin{bmatrix} \frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_1 h}{2}) & 2\beta_1\beta_2\cos(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_2 h}{2}) & p^2\cosh(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_2 h}{2}) \\ -2\beta_1\beta_2\sin(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_1 h}{2}) & p^2\sinh(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\cosh(\frac{\beta_2 h}{2}) & \frac{k_t\beta_2}{2\mu}\sinh(\frac{\beta_2 h}{2}) \\ p^2\cos(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\cosh(\frac{\beta_2 h}{2}) & 0 \\ 0 & p^2\sin(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\sinh(\frac{\beta_2 h}{2}) \end{bmatrix}$$

Case (v):

$$[q_{\alpha\beta}]=\begin{bmatrix} \frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_1 h}{2}) & 2\beta_1\beta_2\cos(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_2 h}{2}) & -p^2\cos(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_2 h}{2}) \\ -2\beta_1\beta_2\sin(\frac{\beta_1 h}{2})+\frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_1 h}{2}) & \frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_1 h}{2}) & p^2\sin(\frac{\beta_2 h}{2})+\frac{k_t\beta_2}{2\mu}\cos(\frac{\beta_2 h}{2}) & \frac{k_t\beta_2}{2\mu}\sin(\frac{\beta_2 h}{2}) \\ p^2\cos(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\cos(\frac{\beta_2 h}{2}) & 0 \\ 0 & p^2\sin(\frac{\beta_1 h}{2}) & 0 & 2(M\pi)^2\sin(\frac{\beta_2 h}{2}) \end{bmatrix}$$

in which β_1 ,β_2 has been written for $|\beta_1|$, $|\beta_2|$ and

$p^2 = 2(M\pi)^2 - \rho\varpi^2/\mu$

References

Akhavan, H., Hashemi, S. H., Taher, H., Alibeigloo, A., & Vahabi, S. (2009). Exact solutions for rectangular Mindlin plates under in-plane loads resting on Pasternak elastic foundation. Part II: Frequency analysis. *Computational Materials Science*, *44*(3), 951-961.

Hanna, N. F., & Leissa, A. W. (1994). A higher order shear deformation theory for the vibration of thick plates. *Journal of Sound and Vibration*, *170*(4), 545-555.

Hasani Baferani, A., Saidi, A. R., & Ehteshami, H. (2011). Accurate solution for free vibration analysis of functionally graded thick rectangular plates resting on elastic foundation. *Composite Structures*, *93*(7), 1842-1853.

Hasheminejad, S. M., & Gheshlaghi, B. (2012). Three-dimensional elastodynamic solution for an arbitrary thick FGM rectangular plate resting on a two parameter viscoelastic foundation. *Composite Structures*, *94*(9), 2746-2755.

Hosseini Hashemi, S., Karimi, M., & Rokni Damavandi Taher, H. (2010). Vibration analysis of rectangular Mindlin plates on elastic foundations and vertically in contact with stationary fluid by the Ritz method. *Ocean Engineering*, *37*(2), 174-185.

Lam, K. Y., Wang, C. M., & He, X. Q. (2000). Canonical exact solutions for Levy-plates on two-parameter foundation using Green's functions. *Engineering Structures*, *22*(4), 364-378.

Levinson, M. (1985). Free vibrations of a simply supported, rectangular plate: an exact elasticity solution. *Journal of Sound and Vibration*, *98*(2), 289-298.

Lim, C. W., Liew, K. M. & Kitipornchai, S. (1998 a). Numerical aspects for free vibration of thick plates Part I: formulation and verification. *Computer Methods in Applied Mechanics and Engineering*, *156*(1), 15-29.

Lim, C. W., Kitipornchai, S., & Liew, K. M. (1998 b). Numerical aspects for free vibration of thick plates Part II: Numerical efficiency and vibration frequencies. *Computer Methods in Applied Mechanics and Engineering*, *156*(1), 31-44.

Malekzadeh, P. (2009). Three-dimensional free vibration analysis of thick functionally graded plates on elastic foundations. *Composite Structures*, *89*(3), 367-373.

Matsunaga, H. (2000). Vibration and stability of thick plates on elastic foundations. *Journal of Engineering Mechanics*, *126*(1), 27-34.

Mindlin, R. D. (1951). Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. *J. of Appl. Mech.*, *18*, 31-38.

Mindlin, R. D., & Deresiewicz, H. (1954). Thickness-Shear and Flexural Vibrations of a Circular Disk. *Journal of Applied Physics*, *25*(10), 1329-1332.

Mindlin, R. D., Schaknow, A. & Deresiewicz, H. (1956). Flexural Vibration of Rectangular Plates. *Journal of Applied Mechanics*, *23*, 430-436.

Omurtag, M. H., Özütok, A., Aköz, A. Y., & OeZCELİKOeRS, Y. U. N. U. S. (1997). Free vibration analysis of Kirchhoff plates resting on elastic foundation by mixed finite element formulation based on Gateaux differential. *International Journal for Numerical Methods in Engineering*, *40*(2), 295-317.

Pasternak, P. L. (1954). On a new method of analysis of an elastic foundation by means of two foundation constants. *Gosudarstvennoe Izdatel'stvo Litearturi po Stroitel'stvu i Arkhitekture, Moscow, USSR (in Russian)*.

Reddy, J. N. (1984). A simple higher-order theory for laminated composite plates. *Journal of applied mechanics*, *51*(4), 745-752.

Reissner, E. (1945). The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics*, *12*(2), 69-77.

Saidi, A. R., Atashipour, S. R., & Jomehzadeh, E. (2009). Reformulation of Navier equations for solving three-dimensional elasticity problems with applications to thick plate analysis. *Acta Mechanica*, *208*(3-4), 227-235.

Sobhy, M. (2013). Buckling and free vibration of exponentially graded sandwich plates resting on elastic foundations under various boundary conditions. *Composite Structures*.99, 76-87.

Srinivas, S., Joga Rao, C. V., & Rao, A. K. (1970). An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates. *Journal of Sound and Vibration*, *12*(2), 187-199.

Tajeddini, V., Ohadi, A., & Sadighi, M. (2011). Three-dimensional free vibration of variable thickness thick circular and annular isotropic and functionally graded plates on Pasternak foundation. *International Journal of Mechanical Sciences*, *53*(4), 300-308.

Timoshenko, S. P., & Goodier, J. N. (1951). *Theory of elasticity*. McGraw-Hill, New York

Timoshenko, S., & Woinowsky-Krieger, S. (1970). *Theory of plates and shells* . New York: McGraw-Hill.

Wen, P. H. (2008). The fundamental solution of Mindlin plates resting on an elastic foundation in the Laplace domain and its applications. *International Journal of Solids and Structures*, *45*(3), 1032-1050.

Winkler, E. (1867). Die Lehre von der Elasticitaet und Festigkeit. *Prag, Dominicus*.

Xiang, Y., Wang, C. M., & Kitipornchai, S. (1994). Exact vibration solution for initially stressed Mindlin plates on Pasternak foundations. *International Journal of Mechanical Sciences*, *36*(4), 311-316.

Zhong, Y., & Yin, J. H. (2008). Free vibration analysis of a plate on foundation with completely free boundary by finite integral transform method. *Mechanics Research Communications*, *35*(4), 268-275.

Zhou, D., Cheung, Y. K., Lo, S. H., & Au, F. T. K. (2004). Three-dimensional vibration analysis of rectangular thick plates on Pasternak foundation. *International Journal for Numerical Methods in Engineering*, *59*(10), 1313-1334.