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State of art for hybrid mixed finite element formulation in non-linear analysis of structures

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A B S T R A C T

Since the late 80's the Structural Analysis Research Group of the Instituto Superior Técnico (IST) has been involved in the development of non-conventional finite element formulations in order to overcome some of the limitations associated with the use of the CFE method and to develop high performance numerical tools for the analysis of structural engineering problems. Several alternative models for the linear and non-linear structural analysis have been developed using hybrid and mixed models techniques. These works are summarized in this paper, in which their past and future applications of this formulation in non-linear analysis of structures are fully detailed.

1. Introduction

"A rational procedure for the construction of assumed stress hybrid finite elements has finally been established. The resulting elements are invariant and have no spurious kinematic deformation modes. They do not suffer from the locking difficulty and the properties of the elements are less sensitive to element distortions. There are additional advantages for formulation of plate and shell elements using the hybrid stress approach. It is easy to separate the rigid body, pure bending and pure membrane stress. I foresee that the assumed stress elements will be very widely adapted in large-scale computer software. I also feel that, in the future, advanced finite element techniques should be incorporated in structural optimization and in computer aided structural design". Quotes from Theodore H. H. Pian in 1984 "Early FEM Pioneers by John Robinson".

In the computational mechanics field, the most popular method for solving non-linear structural problems is the Finite Element Method (Hughes, 2003). This method is still nowadays the most popular numerical technique used in the field of continuum mechanics, due to its explicit physical meaning (Ghali et al., 1997), and its theoretical robustness and efficiency (Clough et al., 1999). In spite of the popularity and computational robustness of the conventional finite element method (CFE), this formulation still presents some drawbacks (Bathe, 1996) and some unfixed problems (Zienkiewicz, 2000). For this reason, several alternative formulations have been proposed over the years, such as: hybrid stress (T.H.H. Pian et al., 1969), hybrid displacement (Veubeke, 1965), Trefftz elements (Jirousek et al., 1977; Ruoff, 1973; Stein, 1973), the extended finite element method (XFEM) (Moes et al., 1999), isogeometric finite elements (Cottrell et al., 2009) and meshfree methods (Liu, 2003). The main limitations associated with the use of CFE formulation can be summarized as follows:

- The solutions are kinematically admissible. As a consequence, in physically non-linear analysis, the solution provided is unsafe from the structural point of view (Reddy, 2004).
- The quality of the results depends strongly on the mesh being used. Good solutions require mesh generators with optimal geometrical algorithms (Frey et al., 2000).

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- The mesh to be adopted depends strongly on the kinematic and static boundary conditions. Special techniques must be applied in order to acquire a stable solution (Piteri, 1999).
- The use of *p*- refinement may be limited, due to the difficulty in constructing the approximation functions. Also for higher degree, the approximation functions of CFE tend to be unstable (Bathe, 1982).
- The distortion of the mesh may strongly influence the results, due to the approximation functions being usually bi-linear and quadratic functions (Cook et al., 2002). In distorted elements, the governing system can only be computed with numerical integration in linear analysis.
- The "Locking" effect can only be avoided either with reduce integration or mixed formulations (Malkus et al., 1978).
- In physically non-linear analysis, the results are controlled either by the stress or deformation fields. Since the CFE only approximates the displacement field, the stress and deformation fields are achieved with a weaker accuracy (Crisfield, 1991).
- For problems with singularities, such as sharp wedges or cracks, the convergence rate does not improve, even with regular elements using high degrees of approximations functions (Hughes, 2003).

Since the late 80s the Structural Analysis Research Group of Instituto Superior Técnico, has been involved in the development of alternatives for hybrid and mixed finite element formulations. Even if these formulations present some advantages, they still have some limitations (Freitas et al., 1999). In recent years, hybrid-mixed stress finite element models have been developed for the static analysis of plane stretching and plate bending problems (Castro et al., 2006; Castro et al., 2001; Pereira et al., 1996a, 2000). After establishing the properties of these alternative formulations, the next challenge was to extend their application to the analysis of problems closer to engineering practice. In a first step, elastoplastic models for the analysis of plane structures have been developed. After that, continuum damage models for the physically non-linear analysis of concrete structures have been studied and implemented (Silva, 2006; Silva, 2002). Recently new hybrid double mixed finite elements are put to the test for static and dynamic non-linear analysis of concrete structures.

2. Introduction to Hybrid and Mixed Formulation

2.1 Main Advantages

The main advantages associated with the use of this type of formulation is the flexibility introduced in the selection of the approximation functions. This fact makes it possible to use functions with special properties that cannot be implemented in the framework of conventional finite element models, such as Legendre polynomials, systems of wavelets and Walsh series. The hybrid-mixed stress models may also lead to quasi-equilibrated solutions which are very suitable for design purposes. It is then possible to define the main advantages of hybrid mixed stress formulations:

- There is a great flexibility in choosing the approximation functions (Freitas et al., 1999).
- The right choice for the degrees of the approximation functions may lead to "quasi-equilibrated" solutions (Arruda, 2008).
- If the approximation functions are orthogonal, the governing system is sparse (Mendes, 2002).
- In the hybrid mixed stress (HMS) model, more specifically the mixed formulations, the deformation field is separated from the stress field. If the degrees are correctly chosen then the shear locking does not occur (Malkus et al., 1978).
- The *p*- refinement is effective and easy to implement, and in linear analysis, all operators of the governing system are computed analytically. No numerical integration is needed in linear analysis (Pereira et al., 1996a).
- For simple geometries the use of macro-elements avoids the use of mesh generators (Pereira, 1993).
- In non-linear analysis, the evolution of the constitutive relation depends directly on the stress field. Since the HMS model directly approximates the stress field, some advantages may be obtained (Lourenço, 2000).

2.2 Early Works

One of the first contributions to the development of hybrid and mixed finite elements is due to (Pian et al., 1969). In this paper, a clear notion of equilibrated or compatible models, based on the use of energy principles, is presented. The hybrid-Trefftz finite element model has been developed by (Jirousek, 1978), following the ideas initially introduced by Treffz (1926). The work developed by (Gallagher, 1965) proved that the finite elements introduced by Turner et al. (1956) and Pian (1964) were identical. In that work the similarity between the Hellinger-Reissner Principle and the Principle of Complementary energy was proved, as long as the homogeneous equilibrium equation is verified. Later, in the work (Washizu, 1982), the term "Hybrid Mixed Finite Element" was coined for the first time. The main conditions governing the convergence of any hybrid mixed, hybrid or mixed finite element in linear analysis is fully presented and demonstrated in the works of (Brezzi et al., 1991; Oden et al., 1983).

2.3 Initial Research Path

In the 90's, two group of authors (Babuska et al., 1994a, 1994b) and (Boroomand et al., 1992; Zienkiewicz et al., 1992a, 1992b) almost separately developed theories that connect the energy norm with mesh refinement using efficient error estimates, for the case of hybrid and mixed formulations Fig. 1.

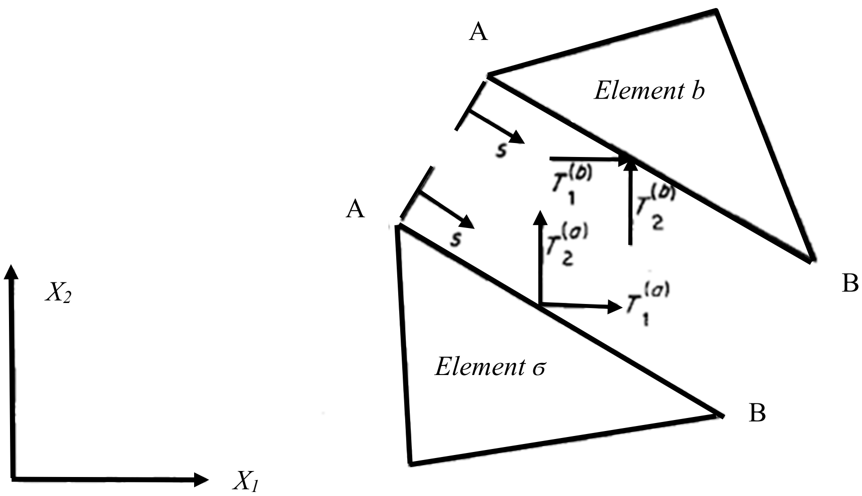


Fig. 1. Equilibrium of boundary tractions along the interelement boundary (T.H.H. Pian et al., 1969).

In Portugal, one of the first contributions concerning the development of hybrid and mixed finite element formulations has been presented by (Almeida et al., 1991). In this work, illustrated in Fig. 2, the most relevant properties of such models have been presented and discussed. The PhD thesis of Almeida (1991) was mainly devoted to the physically non-linear analysis of plane structures using plasticity models. The main goal of this study was the computation of upper and lower bounds for the structure collapse load.

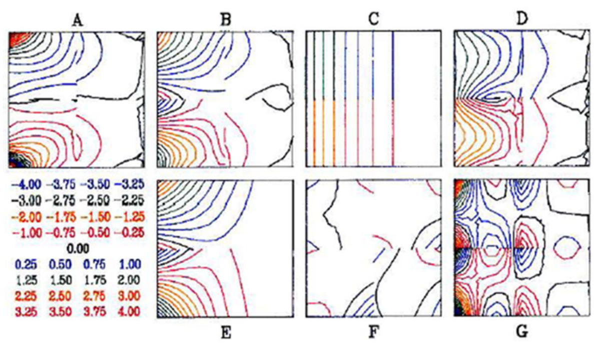


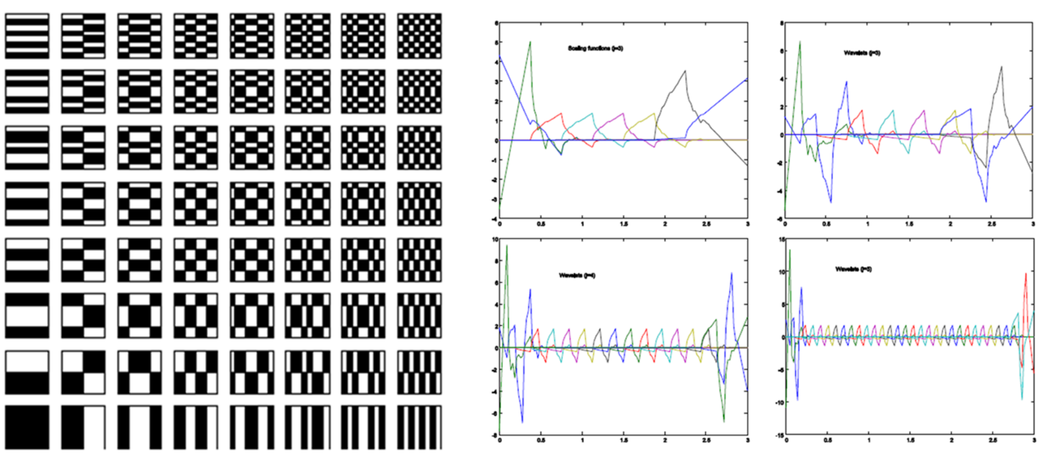
Fig. 2. σ_{xx} stress field in a cantilever (J.P.B.M. Almeida, 1991), after B several degrees of approximations are presented.

An important review of the non-conventional finite element formulations developed by the Structural Analysis Research Group has been presented in (Freitas et al., 1999). In this work, three alternative sets of non-conventional finite elements are discussed: the hybrid, the hybrid-Trefftz and the hybrid-mixed formulations. Taking into consideration the treatment of the domain equations and the conditions used to define the connection between neighbouring elements, two different models may be defined for each formulation: the stress and the displacement models. When using hybrid stress/displacement (HS/HD) models, the stress/displacement fields in the domain of each element and the displacement/applied stress fields on the static/kinematic boundary are independently and simultaneously approximated. The domain approximations are required to locally satisfy equilibrium/compatibility conditions. The boundary approximations are used to enforce, in a weighted residual form, the equilibrium/compatibility conditions at the boundary.

When using hybrid-mixed stress and displacement (HMS/HMD) formulations, none of the fundamental conditions in the domain has to be a *priori* locally satisfied. This fact introduces a great flexibility in the choice of the approximation functions to be used. In this type of formulation, both the stress and the displacement fields are directly approximated in the domain of each finite element. When using the HMS/HMD model, the displacement/applied stress fields are independently approximated at the static/kinematic boundary. The boundary approximations are used to enforce, in a weighted residual form, the equilibrium/compatibility conditions at the boundary. The hybrid-Trefftz models can be viewed as special hybrid formulations. In this case, the approximation functions defined in the domain are required to locally verify all domain conditions: equilibrium, compatibility and constitutive relations. One of the first applications of hybrid stress (HS) elements for the non-linear static analysis of reinforced concrete frames was developed by (Pereira, 1989). The use of hybrid and hybrid-mixed finite element models based on the use of orthonormal Legendre polynomials and trigonometric functions for the analysis of plane structures has been explored in (Pereira, 1993). Another important contribution was given by (Pereira, 1993; Pereira, 1996), studying the bounds and error for adaptive refinement, using hybrid stress and hybrid displacement elements, for linear analysis. The best reference to study and analyse the errors in conventional and non-conventional formulation can be found in the work of (Almeida et al., 2016). As mentioned before, the use of hybrid-mixed models introduces a great flexibility in the choice of the approximation functions. This fact made it possible to use functions with special properties that cannot be used in classical finite element formulations (Castro, 1992; Pereira et al., 1996a, 1996b). Walsh series and systems of wavelets, illustrated in Figure 3, were adopted in the hybrid-mixed models developed for the physically non-linear analysis of the plane structures. These models are fully described in (Castro et al., 1996) and (Castro, 1996).

The combined use of HMS models and Legendre polynomials for the physically non-linear analysis has been discussed in (Lourenço, 2000; Mendes, 2002), as illustrated in Fig. 4 a). The use of Daubechies wavelet orthogonal functions defined in the interval to define approximation bases for HMS models was suggested by (Barbosa, 2002). The success of HMS models for plasticity analysis using Legendre polynomial bases motivated the generalization of these numerical tools to damage mechanics analysis Figure 4 b). The first model combining non-conventional finite elements and damage mechanics models was presented by (Silva, 2002). In this study, the original formulation of the HMS was used to study concrete arch dams, and regularization was performed by using the fracture energy associated with the characteristic length. The next step concerning damage mechanics was accomplished by (Silva, 2006). In this thesis several non-conventional finite element formulations have been tested and different damage models and regularization techniques were applied and compared. One of the most important conclusions was that the original HMS formulation may cause convergence problems in the presence of strong softening, and that hybrid displacement (HD) elements may be more competitive for damage mechanics, since they use stiffness operators in the governing system. In the same thesis, a modification for the HMS was proposed in order to correctly capture the effect of softening when using a governing system ruled by flexibility.

Following the work of Silva (2006), two other research works were developed to test the application of HMS and HD for 3D non-linear static analysis (Garrido et al., 2010; Martins et al., 2010). Even though accurate results have been obtained, the computational cost of the HMS and HD for 3D analysis proved to be very high. This fact produced the need to improve the efficiency of the 3D numerical models, namely by implementing effective parallel computation schemes, this is an ongoing research in hybrid mixed formulation (Antão et al., 2012). The use of wavelets with modified HMS models for damage mechanics has also been implemented (Silva et al., 2010), but the extra computational cost associated with the exponential increase of the number of degrees of freedom made it computationally inefficient.



a) Bidimensional Walsh series b) Unidimensional Wavelets families

Fig. 3 : Approximation functions used in (Castro, 1996).

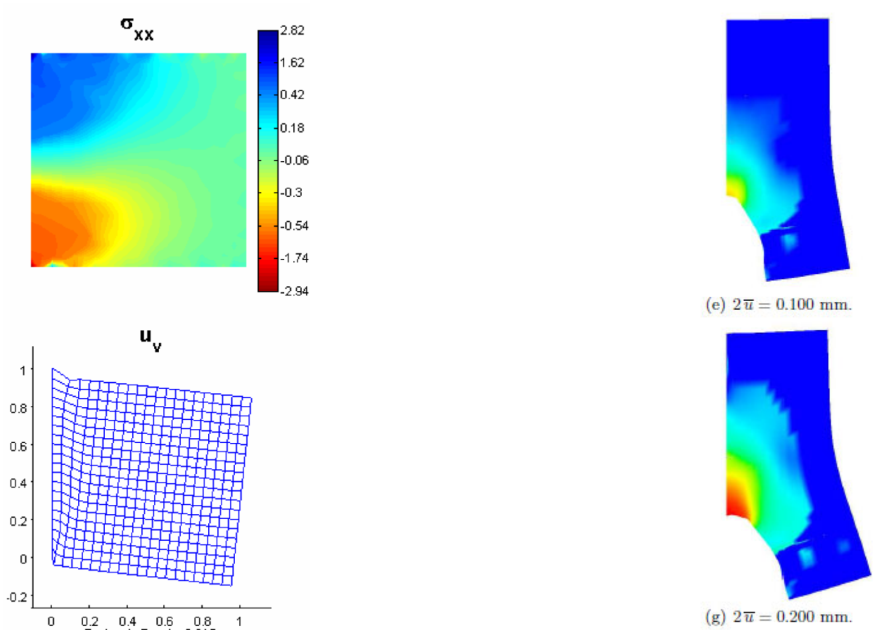
Most of the models mentioned previously were developed for static analysis. To overcome this limitation, (Freitas, 1999) presented a set of non-conventional finite element models for the linear dynamic analysis in the frequency domain. Later on, the displacement model of the hybrid-Trefftz (HT) finite element formulation was applied to the solution of linear static and dynamic problems (Cismasiu, 2000). This type of element is very powerful for the treatment of elastic singularities that cannot be accurately modelled with CFE elements (Fernandes, 1998). The success of this element and the use of plasticity with HM elements, motivated research in the area of dynamic non-linear analysis with plasticity and hybrid-Trefftz finite elements (Wang, 2000). Recently the HT formulation was added to several libraries in structural engineering software, with the possibility of modelling solid and porous media (Ionuţ Dragoş Moldovan et al., 2018; Ionut Dragos Moldovan et al., 2021), for elastodynamics.

2.4 Mixed Time Step Integration

Dynamic analysis using modal superposition (Arruda et al., 2007) and step by step time integration algorithms (M. R. Arruda et al., 2009b, 2012), were applied to frame, plate bending and plane stretching problems. The computation of frequencies and vibration modes using HMS models based on the use of monomials and Legendre polynomials as approximation functions was discussed in (Arruda, 2008; Silva et al., 2005). The step by step time integration algorithms still present some limitations when combined with HMS models (Arruda et al., 2009a). To enhance the use of hybrid and mixed finite element models for dynamic analysis, an alternative algorithm, known here as a Mixed Time Integration technique, has been introduced by (Freitas, 2008; Freitas et al., 2010). The robustness of the HT finite element made possible the development of highly effective models for the elastodynamic analysis of saturated porous media (Moldovan, 2008) and enabled the possibility of modelling hydrated soft tissues (Toma, 2009). Dynamic problems in the civil engineering field were also put to the test with MTI (Arruda et al., 2012; Pina et al., 2004), in which it was proved that for linear analysis the MTI is computational more efficient than conventional step-by-step time integrations methods of the Newmark family, for linear analysis of frames, plates and planes. Most of the above-mentioned analysis deals with linear elastic problems. The algorithm suggested by (Freitas, 2008) had to be generalized in order to cope with physically non-linear analysis (Arruda et al., 2010b). The new technique has been tested in the solution of non-linear frame problems (Arruda et al., 2010a; Arruda et al., 2015) and its efficiency assessed and compared with some classical SSTI techniques (M. R. Arruda et al., 2009c). For frame elements the HMS was also competitive with conventional FE, even with low degrees of approximation (Arruda et al., 2021) for static and dynamic analysis.

2.5 Bond-Slip in Concrete Structures

Recently, a new mixed-mixed finite element model has been introduced by (Pina, 2009), in order to model the bond-slip effect on reinforced concrete frame structures. This FE formulations is based on a modified Bernoulli beam theory, which was developed with the purpose of representing the bond-slip of rebar in concrete structures, more precisely the corroded reinforcement. It included crack distribution due to the rebars being embedded in the mesh discretization, independently of the concrete beam. It is safe to state that the advantages associated with the use of hybrid and mixed formulations are far from being completely explored. This fact justifies the research effort devoted to this type of finite elements and the growing interest in their application to the solution of engineering real-life problems. Later on this same formulation was adopted in an enhanced version of hybrid mixed formulation in the work of (Luz, 2013), in which a full 2D behaviour was now able to be fully simulated.

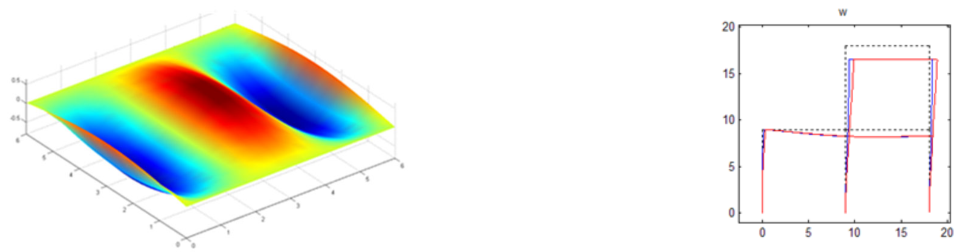


a) Plastic analysis using Legendre polynomials (Mendes, 2002) b) Tension damage (Silva, 2006)

Fig. 4 . Plastic stress and damage distribution using Legendre polynomials

2.6 Geometrically Non-Linear Analysis

Recently, the non-conventional finite element models have been applied to geometric non-linear analysis. The first application of HS models in the computation of stresses and displacements in geometrically non-linear analysis was presented by (Correia et al., 2007). Later on, (Arruda et al., 2015; Arruda, 2009) used HMS models for the computation of critical loads and buckling modes for frame and plate structures. The use of HMS with a geometric exact beam theory was developed by (Santos, 2009), in which a dual analysis with frame structures was performed to compute upper and lower bounds for critical loads Fig. 5 a). New tests concerning linearized stability analysis of frame and plane structures have been performed by (Arruda et al., 2015; Arruda, 2009), in which great advantages have been portrait using *p*- refinement Fig. 5 b). This success is mainly due to the use of HMS finite elements, in which great information is provided by the stress field during the Eigen value analysis.



a) Plane buckling mode b) Frame 2nd order deformed shape

Fig. 5. Linearized stability analysis with HMS.

3. A new finite element

3.1 Main contribution

A new contribution from IST (Arruda, 2011; Arruda et al., 2013; Arruda et al., 2013) corresponds to the proposal of a new Hybrid Mixed Stress element, that considers an independent approximation of the stress, the displacement and also the strain field in the domain of each element. This element will be referred to as the Hybrid Double Mixed Stress element (HDMS or 4f-HMS). Contrary to the HMS, the governing system is ruled by stiffness, which makes it ideal to use in static non-linear analysis based on the use of softening models. This model tends to overcome the difficulties of the HMS in the presence of strong softening. In these initial works static and dynamic non-linear analyses are performed for plane structures.

3.2 Applications

The work reported in (Arruda et al., 2011a) presents a generalization of the hybrid-mixed stress model that allows dynamic non-linear analysis of frame and plane structures. The model is called hybrid because both the stresses in the domain and the displacements at the static boundary (which includes the boundary between elements) are approximated simultaneously. It is called mixed because both the stresses and the displacements in the domain are directly approximated. All the fundamental conditions are imposed using a weighted residual approach designed to ensure that the discrete model embodies all the relevant properties presented by the continuum it represents, namely static-kinematic duality and elastic reciprocity. Due to the type of enforcement followed for the equilibrium and compatibility conditions in the domain and because the connection between elements is ensured by the weighted residual enforcement of the equilibrium conditions at the common boundary, the presented formulation is considered to lead to a stress model (Arruda et al., 2011a, 2011b, 2011c, 2013). This element tends to overcome the known difficulties when using damage mechanics with HMS (Silva, 2006; Silva, 2002), for governing systems based on flexibilities. The finite element models presented in (Arruda, 2011) are based on the use of orthonormal Legendre polynomials as approximation functions. The properties of these functions allows for definition of analytical closed form solutions for the computation of all structural operators in linear analysis. Numerical integration schemes are thus completely avoided. The numerical stability associated with the use of Legendre polynomial bases enables the use of macro-element meshes where the definition of highly effective *p*-adaptive refinement procedures is simplified. A detailed presentation of these functions can be found in (Arruda, 2008; Mendes et al., 2009). The work of (Arruda, 2011) follows the path of the initial research reported by (M. C. Silva, 2006; M. J. V. Silva, 2002) regarding the application of hybrid mixed formulations for the statically physically non-linear analysis of concrete structures using damage mechanics. Therefore the Hybrid Mixed (HM) formulation is studied to solve some of the problems of CFE, and the MTI algorithm is used to overcome the main drawbacks that appear in the classical step by step time integration (SSTI) using a family of Newmark methods in linear and non-linear analysis. Since in all numerical examples the imposed loads caused only small displacements, it is wise to adopt geometrical linear analysis in this work.

3.3 Hybrid Mixed Stress Finite Element

The hybrid-mixed stress model (HMS) considers two independent approximations for the stress and the displacement fields in the domain of each element, expressed as follows:

$$\begin{aligned} s &= s(x,t) = S_V(x)X_V(t) = S_V X_V \quad in \quad V \\ u &= u(x,t) = U_V(x)q_V(t) = U_V q_V \quad in \quad V \end{aligned} \tag{1}$$

Matrices S_V and U_V collect together the approximation functions and vectors X_V and q_V list the corresponding weights. The components of matrices S_V and U_V depend on the space variables and the components of vectors X_V and q_V depend on the time variable. The displacement field along the static boundary (which includes the boundaries between neighbouring elements) is also approximated, as expressed in equation (2).

$$u = u(x,t) = U_\Gamma(x)q_\Gamma(t) = U_\Gamma q_\Gamma \quad in \quad \Gamma_\sigma \tag{2}$$

Matrix U_Γ list the approximation functions and vector q_Γ the corresponding weights. As before, the components of matrix U_Γ depend on the space variables and the components of vector q_Γ depend on the time variable. The equilibrium condition is enforced in the following weighted residual form (Pereira et al., 1996b):

$$\int U_V^t\left(Ds + b - c\dot{u} - m_\rho \ddot{u}\right)dV = 0 \tag{3}$$

It is possible to rewrite equation (3) in the form:

$$A_V^t X_V - \mathbb{C}\dot{q}_V - \mathbb{M}\ddot{q}_V = -Q_V \tag{4}$$

with

$$\mathbb{M} = \int U_V^t m_\rho U_V dV \quad \mathbb{C} = \int U_V^t c U_V dV \quad A_V^t = \int U_V^t D S_V dV \quad Q_V = \int U_V^t b dV \tag{5}$$

The generalized mass matrix, $\mathbb{M}$, takes into account the oscillating mass and the operator $\mathbb{C}$ accounts for damping. Matrix A_V^t corresponds to the equilibrium operator and vector Q_V collects the components of the generalized body forces. The equilibrium conditions at the boundary, the compatibility and the elasticity relations in the domain are also imposed in a weighted residual form. For the equilibrium conditions at the static boundary, no inertia forces are considered and the weighted residual enforcement of these conditions may be expressed by:

$$\int U_\Gamma(\mathcal{N}s - t_\sigma)d\Gamma = 0 \tag{6}$$

Since X_V only depends on the time variable it is possible to obtain the equilibrium conditions at the boundary for the discrete model, in the following format;

$$A_\Gamma^t X = Q_\Gamma \tag{7}$$

with

$$A_\Gamma^t = \int U_\Gamma^t(\mathcal{N}S_V)d\Gamma_\sigma \qquad Q_\Gamma = \int U_\Gamma^t t_\sigma d\Gamma_\sigma \tag{8}$$

Matrix A_Γ^t corresponds to the equilibrium operator at the boundary, this structural operator is also used to enforce the average equilibrium between neighbouring elements. Vector Q_Γ, collects the components of the generalized boundary forces applied at the static boundary. The compatibility conditions in the domain are enforced using the following weighted residual approach:

$$\int S_V^t(e - D^* u)dV = 0 \tag{9}$$

To mobilize boundary terms, the second member of Equation (10), is now integrated by parts, leading to:

$$\int S_V^t D^* u dV = \int (\mathcal{N}S_V)^t u d\Gamma_\sigma + \int (\mathcal{N}S_V)^t u d\Gamma_u - \int (DS_V)^t u dV \tag{10}$$

The compatibility conditions for the discrete model may then be written as follows:

$$\mathbb{e} = -A_V q_V + A_\Gamma q_\Gamma + \bar{u}_\Gamma \tag{11}$$

with

$$\mathbb{e} = \int S_V^t e dV \quad A_V = \int (DS_V)^t U_V dV \quad A_\Gamma = \int (\mathcal{N}S_V)^t U_\Gamma d\Gamma_\sigma \quad \bar{u}_\Gamma = \int (\mathcal{N}S_V)^t \bar{u} d\Gamma_u \tag{12}$$

The constitutive law is also enforced in the weighted residual form presented in equation (13).

$$\int S_V^t(e - fs)dV = 0 \tag{13}$$

The elasticity condition for the discrete model may be written in the following format:

$$\mathbb{e} = \mathbb{F}X_V \tag{14}$$

with

$$\mathbb{F} = \int S_V^t f S_V dV \tag{15}$$

The elementary governing system, Eq. (16) is obtained by combining the equilibrium, compatibility and elasticity conditions in the discrete model. Note that when physically non-linear behaviour is considered, the only operator to be modified corresponds to the generalized flexibility matrix, $\mathbb{F}$. The remaining operators will remain the same for both linear and non-linear analysis, and the mathematical form of these are fully detail in the work of (M. R. T. Arruda, 2011).

$$\underbrace{\begin{bmatrix} \mathbb{F} & -A_\Gamma & A_V \\ -A_\Gamma^t & 0 & 0 \\ A_V^t & 0 & 0 \end{bmatrix}}_{\mathcal{F}} \underbrace{\begin{Bmatrix} X_V \\ q_\Gamma \\ q_V \end{Bmatrix}}_{x} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{Bmatrix} \dot{X}_V \\ \dot{q}_\Gamma \\ \dot{q}_V \end{Bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{M} \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{Bmatrix} \ddot{X}_V \\ \ddot{q}_\Gamma \\ \ddot{q}_V \end{Bmatrix}}_{\ddot{x}} = \underbrace{\begin{Bmatrix} \bar{u}_\Gamma \\ -Q_\Gamma \\ -Q_V \end{Bmatrix}}_{\mathcal{Q}(t)}$$
$$\mathcal{F}x - \mathcal{C}\dot{x} - \mathcal{M}\ddot{x} = \mathcal{Q}(t) \tag{16}$$

3.4 Double Mixed Stress Finite Element

The hybrid double mixed stress (HDMS) model considers an extra approximation in the domain of each finite element. The strain field is independently approximated in the domain of each finite element in the form:

$$e = E_V \epsilon_V \quad in \quad V \tag{17}$$

Matrix E_V collects the approximation functions and vector ϵ_V lists the corresponding weights (17). The other approximations designed by the hybrid mixed stress model are also considered. The new model defines an independent approximation for the stress, strain and displacements field in the domain of each element. The displacements at the static boundary are also directly approximated. Using the weighted residual enforcement of the compatibility condition in the domain defined in equation (13), integrating the second member by parts to mobilize the boundary terms, replacing the approximation defined for the displacement field both in the domain (3,6) and at the static boundary and (6) for the strain field in the domain (14) it is possible to obtain:

$$\int S_V^t e dV = \int S_V^t D^* u dV \tag{18}$$

The compatibility condition for the discrete model may be expressed as:

$$B_V^t \epsilon_V = -A_V q_V + A_\Gamma q_\Gamma + \bar{u}_\Gamma \tag{19}$$

with

$$B_V^t = \int S_V^t E_V \, dV \tag{20}$$

The treatment of the constitutive relation is now substantially different. The local constitutive relation is defined for this purpose in the stiffness format. This is one of the main advantages associated with the use of this new hybrid-mixed stress model: the constitutive relation is written in the stiffness format which is quite convenient when damage models are incorporated in the model to simulate material physically non-linear behaviour. The constitutive relations are enforced to a weighted residual form as follows:

$$\int E_V^t(s - ke)dV = 0 \tag{21}$$

Replacing in Eq. (21) the approximations defined for the stress and strain fields, it is possible to obtain:

$$B_V X_V = \mathbb{K}\epsilon_V \tag{22}$$

with

$$\mathbb{K} = \int E_V^t k E_V dV \tag{23}$$

The governing system (24) is obtained by combining the equilibrium, compatibility and elasticity conditions in the discrete model. Note again that when physically non-linear behaviour is considered, the only operator to be modified corresponds to the generalized stiffness matrix, $\mathbb{K}$. The remaining operators will remain the same for both linear and non linear analysis.

$$\underbrace{\begin{bmatrix} -\mathbb{K} & B_V & 0 & 0 \\ B_V^t & 0 & -A_\Gamma & A_V \\ 0 & -A_\Gamma^t & 0 & 0 \\ 0 & A_V^t & 0 & 0 \end{bmatrix}}_{\mathcal{K}} \underbrace{\begin{Bmatrix} \epsilon_V \\ X_V \\ q_\Gamma \\ q_V \end{Bmatrix}}_{x} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{Bmatrix} \dot{\epsilon}_V \\ \dot{X}_V \\ \dot{q}_\Gamma \\ \dot{q}_V \end{Bmatrix}}_{\dot{x}} - \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{M} \end{bmatrix}}_{\mathcal{M}} \underbrace{\begin{Bmatrix} \ddot{\epsilon}_V \\ \ddot{X}_V \\ \ddot{q}_\Gamma \\ \ddot{q}_V \end{Bmatrix}}_{\ddot{x}} = \underbrace{\begin{Bmatrix} 0 \\ \bar{u}_\Gamma \\ -Q_\Gamma \\ -Q_V \end{Bmatrix}}_{Q(t)} \tag{24}$$

$$\mathcal{K}x - \mathcal{C}\dot{x} - \mathcal{M}\ddot{x} = Q(t)$$

3.5 *Initial Non-Linear Analysis*

At the moment this new element was been fully tested using damage mechanics for static (Arruda et al., 2011a, 2011c, 2013) and dynamic (Arruda et al., 2013) non-linear analysis. To validate the model and to illustrate its potential, several numerical tests are presented and discussed in (Arruda et al., 2011a, 2011c, 2013) Figure 6. The results obtained are directly compared with experimental results and with conventional finite element numerical solutions other numerical solutions computed with the classical displacement finite element formulation. The examples presented show clearly that the model is stable, robust and leads to accurate results. Effective *p*- and *h*- refinement procedures can also be easily implemented. Since it is possible the hierarchy refinement without changing the mesh, macro-elements of large dimensions can be used, which greatly simplifies the operations of post-processing. In the HMS model, the discretizations can lead to systems with a relatively large number of degrees of freedom. However, such matrix are always very sparse, the use of algorithms specially designed for the storage and processing of sparse systems of large dimensions to ensures the effectiveness of any numerical calculation process. The relationship between the degrees of the polynomials used for approximation of the stress fields and degrees of the polynomials associated with the displacement field should be carefully established to avoid the appearance of spurious modes, which are manifested through the existence of dependencies in the governing system. Since HDMS model does not verify neither equilibrium nor compatibility conditions, nothing can be concluded for the upper or lower bond of the solutions calculated.

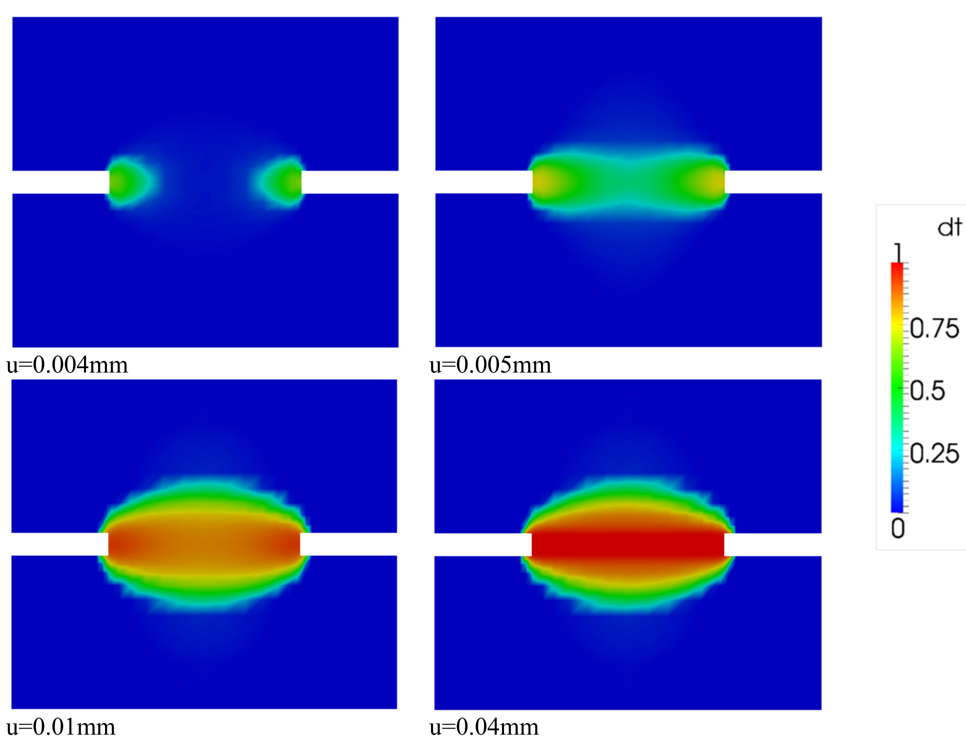


Fig. 6. Tension Damage Evolution (M. R. Arruda et al., 2011a).

4. Remarks and Further Developments

4.1 Remarks

This new hybrid mixed stress finite element model was used for the physically non-linear analysis of two-dimensional concrete structures. Its origin can be traced back to the original HMS model. The main difference when compared to the original version is the direct and independent approximation of the strain field. The main objective of this method is to overcome the difficulties associated with the use of HMS finite elements for damage mechanics analysis, as described in the work of (Silva et al., 2005). These objectives are all achieved and a new stable, robust and computationally effective numerical tool has been proposed. The main computational cost associated with the use of the HDMS model is the computation of the non-linear stiffness operator. For this reason, three different strategies have been implemented and tested for both static and dynamic analysis. For static analysis, the first implementation (the stiffness matrix is updated in every iteration) presented some convergence problems when a single finite element becomes completely damaged $(d \simeq 1.0)$. To overcome this problem, the maximum value of damage allowed for the construction of the stiffness operator was controlled. The examples presented in this work (Arruda, 2011) showed that the second implementation (the stiffness matrix is updated only at the beginning of each load increment) is the most successful in terms of computational cost. The use of HDMS models ensures the computation of accurate damage distributions even for low degrees of approximation. If the degree used to define the approximation bases is high enough, the quality of the solution is less affected by mesh distortion. Macro-element meshes and effective *p*-refinement procedures can be used in the analyses and the overall computational cost associated with the use of these models is usually very competitive when compared to the use of CFE implementations. This advantage is due to the fact that the strain field is directly approximated, which makes it ideal to use with damage mechanics. For the dynamic analysis, simple examples have been performed with cyclic loads and prescribed displacements. These were all academic tests but they are useful to assess the numerical stability and the computational efficiency of the HDMS models in non-linear analysis. In the presence of strong softening, the existence of numerical noise may cause some convergence problems along the iterative process. For dynamic analysis, only the second and third (the stiffness matrix is never updated) implementations ensure the convergence of the iterative process (M. R. T. Arruda et al., 2013). Using full 2D plane structures with reinforced concrete structures was tested by Miguel Luz et al. (2013), but with the particular case of using only perfect bond-slip between the steel rebars and the concrete.

4.2 Further Developments

The results presented in (Arruda, 2011) show that this model may be competitive when compared to the classical numerical tools. Further studies are however necessary to fully exploit the advantages associated with its use. The generalization of the model to make possible its application to the solution of other kinds of structural engineering problems it is also envisaged. It is expected that in the near future the following topics will be addressed:

- Assessment of the convergence properties associated with the use of the new HDMS for static linear analysis. A stability analysis of this numerical model must be performed based on the verification of the Babuska-Brezzi (LBB) (Gois, 2009) conditions.
- Use of the HDMS models for the static and dynamic analysis of reinforced concrete frame elements, to evaluate their advantages when compared to other numerical methods.
- Test of HDMS models based on the use of orthotropic damage models, both for static and dynamic analysis in order to try to overcome the convergence problems related to the transition from tension to compression states.
- Combine damage and plasticity since the compression concrete behaviour is closer to pure plasticity.
- To combine fracture mechanics models with hybrid and mixed formulations, trying to take advantage of the *p*-refinement to localize crack paths, using “weak“ and “strong” discontinuities.
- Using the new theory of crack location, it is possible to exploit the advantages associated with the use of HDMS models to study the transition from damage mechanics to fracture mechanics, it being possible to model the initial diffuse micro-cracking of concrete and the coalescent macro-cracks that appear as damage develops.
- Analysis of two-dimensional reinforced concrete structures in which bond slip may occur.
- Generalization of the model in order to allow for three-dimensional analysis with solid elements.
- Parallel implementation of the HDMS model in order to reduce the computational cost, taking advantage of the sparse structure of the governing system using Open-MPI and GPU.
- Use of alternative functions to define the approximations required by the MTI technique in order to control the level of numerical damping.

For several years the hybrid and mixed finite elements have been fully tested, but still many questions need to be answered. It is not possible to state if these elements will ever completely replace the CFE for non-linear analysis, but as Professor Pian has pointed out, a promising future is expected for this formulation.

5. Dedication

This paper is dedicated to all the researchers from the Department of Civil Engineering in Instituto Superior Técnico, that in the last 40 years have pursue the difficult task on testing and comparing non-conventional formulation with the standard finite elements. In addition, special thanks are given to Professor Freitas, whose keenness allowed this research path to grow, and be used in more advance analyses.

References

Almeida, J. P. B. M., & Maunder, E.A.W. (2016). *Equilibrium Finite Element Formulations*: John Wiley & Sons.

Almeida, J.P.B.M, & Freitas, J.A.T. (1991). Alternative approach to the formulation of hybrid equilibrium finite elements. *Computer & Structures, 40*(4), 1043-1047.

Almeida, J.P.B.M. (1991). *Modelos de Elementos Finitos para a Análise Elastoplástica.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Antão, A. N., Vicente da Silva, M., Guerra, N., & Delgado, R. (2012). An upper bound-based solution for the shape factors of bearing capacity of footings under drained conditions using a parallelized mixed f.e. formulation with quadratic velocity fields. *Computers and Geotechnics, 41*, 23-35. doi: https://doi.org/10.1016/j.compgeo.2011.11.003

Arruda, M. R. T., & Castro, L. M. S. (2021). Non-linear dynamic analysis of reinforced concrete structures with hybrid mixed stress finite elements. *Advances in Engineering Software, 153*, 102965. doi: https://doi.org/10.1016/j.advengsoft.2020.102965

Arruda, M.R. (2008). *Análise Dinâmica de Estruturas com Elementos Finitos Híbridos-Mistos de Tensão.* (M.Sc Thesis), Instituto Superior Técnico, Lisboa.

Arruda, M.R. (2011). *Static and Dynamic Analysis of Concrete Structures Using Damage Mechanics.* (Ph.D. Thesis), Instituto Superior Técnico, Lisbon.

Arruda, M.R., & Castro, L.M.S.S. (2007). *Análise Dinâmica de Lajes de Reissner-Mindlin com Modelos Híbridos-Mistos de Tensão*. Paper presented at the Congresso de Métodos Numéricos em Engenharia Porto.

Arruda, M.R., & Castro, L.M.S.S. (2009a). *Implementação de Modelos Híbridos-Mistos de Tensão para Análise Dinâmica no Domínio do Tempo*. Paper presented at the Congreso de Métodos Numéricos en Ingeniería, Barcelona.

Arruda, M.R., & Castro, L.M.S.S. (2009b). Modelos Híbridos-Mistos de Tensão para a Análise Dinâmica no Domínio do Tempo. *Revista Portuguesa de Engenharia de Estruturas, 5*(2), 13-19.

Arruda, M.R., & Castro, L.M.S.S. (2009c). *Modelos Híbridos-Mistos de Tensão para Análise Dinâmica Fisicamente Não-Linear de Pórticos de Betão Armado*. Paper presented at the Congreso de Métodos Numéricos en Ingeniería, Barcelona.

Arruda, M.R., & Castro, L.M.S.S. (2010a). *Hybrid-Mixed Stress Finite Element Models for the Dynamic Analysis of Reinforced Concrete Frame Structures*. Paper presented at the Proceedings of the Tenth International Conference on Computational Structures Technology, Valencia.

Arruda, M.R., & Castro, L.M.S.S. (2010b). *Time Integration Procedures with Hybrid-Mixed Stress Finite Elements*. Paper presented at the Proceedings of the Tenth International Conference on Computational Structures Technology, Valencia.

Arruda, M.R., & Castro, L.M.S.S. (2011a). *Continuum damage model using hybrid mixed formulation*. Paper presented at the CORAN, Coimbra.

Arruda, M.R., & Castro, L.M.S.S. (2011b). *Hybrid mixed formulation in continuum damage mechanics*. Paper presented at the CFRAC, Barcelona.

Arruda, M.R., & Castro, L.M.S.S. (2011c). *Modelos de Dano Contínuo com Elementos Finitos Híbridos-Mistos de Tensão*. Paper presented at the CMNE, Coimbra.

Arruda, M.R., & Castro, L.M.S.S. (2012). Structural dynamic analysis using hybrid and mixed finite element models. *Finite Elements in Analysis and Design, 57*, 43-54.

Arruda, M.R., & Castro, L.M.S.S. (2013). A new hybrid-mixed stress model for the analysis of concrete structures using damage mechanics. *Computer & Structures, 115*, 23-44.

Arruda, M.R.T. (2011). *Static and Dynamic Analysis of Concrete Structures Using Damage Mechancis.* (Phd), University of Lisbon in Instituto Superior Técnico, Lisbon.

Arruda, M.R.T., & Moldovan, D.I. (2015). On a mixed time integration procedure for non-linear structural dynamics. *Engineering Computations, 32*(2), 329-369. doi: 10.1108/EC-05-2013-0136

Arruda, Mário R. T., & Castro, Luís Manuel Santos. (2013). Static and dynamic physically non-linear analysis of concrete structures using a hybrid mixed finite element model. *Advances in Engineering Software, 65*(0), 112-131. doi: http://dx.doi.org/10.1016/j.advengsoft.2013.06.002

Arruda, P. F. T., Arruda, M. R. T., & Castro, L. M. Santos. (2015). Computation of critical loads and buckling modes using hybrid-mixed stress finite element models. *Computers & Structures, 154*, 72-90. doi: https://doi.org/10.1016/j.compstruc.2015.02.012

Arruda, P.F.T. (2009). *Análise de estabilidade de estruturas com elementos finitos híbridos-mistos de tensão.* (M.Sc Thesis), Instituto Superior Técnico, Lisbon.

Babuska, I., Strouboulis, T., Upadhyay, C.S., Gangaraj, S.K., & Copps, K. (1994a). An objective criterion for acessing the reliability of a-postriori error estimates in finite element computation. *USACM Bulletin, 4*(1), 4-15.

Babuska, I., Strouboulis, T., Upadhyay, C.S., Gangaraj, S.K., & Copps, K. (1994b). Validation of a-postriori error estimator by numerical approach. *International Journal of the Numerical Methods in Engineering, 37*(1), 1073–1123.

Barbosa, A.R. (2002). *Wavelets no Intervalo em Elementos Finitos.* (M.Sc Thesis), Instituto Superior Técnico, Lisbon.

Bathe, K.J. (1982). *Finite Element Procedures In Engineering Analysis*: Prentice-Hall Englewood Cliffs.

Bathe, K.J. (1996). *Finite Element Procedures In Engineering Analysis*: Prentice-Hall Englewood Cliffs.

Boroomand, B., & Zienkiewicz, O.C. (1992). The superconvergent patch recovery and a-posteriori error estimates; part i: The recovery thechnique; part ii: Error estimates and adaptativity. *International Journal of the Numerical Methods in Engineering, 33*, Part I: 1331–1365; Part II: 1365–1382.

Brezzi, F., & Fortin, M. (1991). *Mixed and Hybrid Mixed Finite Element Methods*. New-York: Springer.

Castro, L.M.S.S. (1992). *Interpolação de Walsh em Problemas de Elasticidade Plana.* (M.Sc Thesis), Instituto Superior Técnico, Lisbon.

Castro, L.M.S.S. (1996). *Wavelets e Séries de Walsh em Elementos Finitos.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Castro, L.M.S.S., & Barbosa, A.R. (2006). Implementation of an Hybrid-Mixed Stress Model Based on the use of Wavelets. *Computer & Structures, 84*(10-11), 718-731.

Castro, L.M.S.S., & Freitas, J.A.T. (1996). *Hybrid-Mixed Finite Element Elastoplastic Analysis Based on Walsh and Wavelet Interpolation*. Paper presented at the Congresso de Metodos Numéricos em Engenharia, Barcelona.

Castro, L.M.S.S., & Freitas, J.A.T. (2001). Wavelets in Hybrid-Mixed Stress Elements. *Computer Methods in Applied Mechanics and Engineering, 190*(31), 3977-3998.

Cismasiu, C. (2000). *The hybrid-Trefftz displacement element for static and dynamic structural analysis problems.* (Ph.D Thesis), Instituto Superior Técnico, Lisbon.

Clough, R. W, & Wilson, E. (1999). *Early Finite Element Research at Berkeley.* Paper presented at the National Conference on Computational Mechanics.

Cook, R.D, Malkus, D.S. , Plesha, M.E., & R.J., Witt. (2002). *Concepts and Applications of Finite Element Analysis*. USA: John Wiley and Sons Inc.

Correia, Antonio A., Almeida, J Pacheco., & Pinho, Rui. (2007). *Comparação entre Formulações de Equilibrio e Deslocamentos na Análise Não Linear de Estruturas Porticadas de Betão Armado.* Paper presented at the Congreso de Sismologia e Engenharia Sísmica, Porto.

Cottrell, J.A., Hughes, T.J.R., & Bazilevs, Y. (2009). *Isogeometric analysis: Toward Integration of CAD and FEA*. Chichester: Wiley.

Crisfield, M.A. (1991). *Non-linear Finite Element Analysis of Solids and Structures* (Vol. Volume 1 - Essentials). Chichester: John Wiley & Sons.

Fernandes, C.M.T.T. (1998). *Utilização e Desenvolvimento de uma Formulação Indirecta de Trefftz na Análise de Lajes Finas.* (M.Sc Thesis), Instituto Superior Técnico, Lisboa.

Freitas, J.A.T. (1999). Hybrid Finite Element Formulation for Elastodynamic Analysis in the Frequency Domain. *International Journal of Solids and Structures, 36*(13), 1883-1923.

Freitas, J.A.T. (2008). Mixed Finite Element Solution of Time-Dependent Problems. *Computer Methods in Applied Mechanics and Engineering, 197*(45-48), 3657-3678.

Freitas, J.A.T., Almeida, J.P.B.M, & Pereira, E.M.B.R. (1999). Non-conventional Formulations for the Finite Element Method. *Computational Mechanics, 23*(1), 488-501.

Freitas, J.A.T., Moldovan, I.D., & Toma, M. (2010). Mixed and hybrid stress elements for biphasic media. *Computers & Structures, 88*(23-24), 1286-1299.

Frey, P. J., & George, P. L. (2000). *Mesh Generation - application to finite elements*. United Kingdom: HERMES Science Publishing.

Gallagher, R.H. (1965). Comments on the derivation of element stiffness matrices by assumed stress distributions. *A.I.A.A. Journal, 3*(1), 186–187.

Garrido, C.S.R., & Castro, L.M.S.S. (2010). *Hybrid mixed stress finite element model for the physically nonlinear analysis of three-dimensional concrete structures*. Paper presented at the Civil-Comp Press 2010, Valencia.

Ghali, A., & Nevillle, A.M. (1997). *Structural Analysis: A unified Classic and Matrix Approach*. London: E & FN Spon.

Gois, W. (2009). *Elementos Finitos Híbridos-Mistos de Tensão com Enriquecimento Nodal.* (Ph.D. Thesis), Escola de Engenharia de São Carlos, Universidade de São Paulo.

Hughes, T.J.R. (2003). *The Finite Element Method", Linear Static and Dynamic Finite Element Analysis*: Dover.

Jirousek, J. (1978). Basis for development of large finite elements locally satisfiying all field equations. *Computer Methods in Applied Mechanics and Engineering, 14*, 65-92.

Jirousek, J., & Leon, N. (1977). A Powerful Finite Element for Plate Bending. *Computer Methods in Applied Mechanics and Engineering, 12*, 77-96.

Liu, G.R. (2003). *Mesh Free Methods*. USA: CRC Press.

Lourenço, M. (2000). *Elementos Finitos de Tensão - Aplicação à Análise Elastoplástica de Estruturas Laminares Planas.* (M.Sc Thesis), Instituto Superior Técnico, Lisboa.

Luz, M. (2013). *Modelos híbridos-mistos de tensão para a análise de estruturas de betão armado.* (MSc), University of Lisbon IST, Portugal.

Malkus, D.S., & Hughes, T.J.R. (1978). Mixed Finite Element Methods - Reduced and Selective Integration Techniques: a Unification of Concepts. *Computer Methods in Applied Mechanics and Engineering, 1*(15), 63-81.

Martins, J.M.A., & Castro, L.M.S.S. (2010). *Hybrid displacement finite element model for the physically nonlinear analysis of three-dimensional concrete structures*. Paper presented at the Civil-Comp Press 2010, Valencia.

Mendes, L.A.M. (2002). *Modelos de Elementos Finitos Híbridos Mistos de Tensão na Análise Elastoplástica de Estruturas Laminares Planas.* (M.Sc Thesis), Instituto Superior Técnico, Lisboa.

Mendes, L.A.M., & Castro, L.M.S.S. (2009). Hybrid-mixed stress finite element models in elastoplastic analysis. *Finite Elements in Analysis and Design, 45*(12), 863-875.

Miguel Luz, Arruda, M., & Castro, L. (2013). HYBRID-MIXED STRESS MODELS FOR THE ANALYSIS OF REINFORCED CONCRETE STRUCTURES. DeCivil: Instituto Superior Técnico.

Moes, N., Dolbow, J., & Belytschko, T. (1999). A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering, 46*, 131-150.

Moldovan, I.D. (2008). *Hybrid-Trefftz Finite Elements for Elastodynamic Analysis of Saturated Porous Media.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Moldovan, Ionuţ Dragoş, & Cismaşiu, Ildi. (2018). FreeHyTE: a hybrid-Trefftz finite element platform. *Advances in Engineering Software, 121*, 98-119. doi: https://doi.org/10.1016/j.advengsoft.2018.03.014

Moldovan, Ionut Dragos, Climent, Natàlia, Bendea, Elena Daniela, Cismasiu, Ildi, & Gomes Correia, António. (2021). A hybrid-Trefftz finite element platform for solid and porous elastodynamics. *Engineering Analysis with Boundary Elements, 124*, 155-173. doi: https://doi.org/10.1016/j.enganabound.2020.12.014

Oden, J. T., & Carey, G.F. (1983). *Finite Elements Mathematical Aspects* (Vol. IV). New Jersey: Prentice-Hall.

Pereira, E.M.B.R. (1989). *Análise Fisicamente Não Linear de Porticos Tridimensionais de Betão Armado.* (M.Sc Thesis), Instituto Superior Técnico, Lisboa.

Pereira, E.M.B.R. (1993). *Elementos Finitos de Tensão, Aplicação à Análise Elástica de Estruturas.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Pereira, E.M.B.R., & Freitas, J.A.T. (1996a). A Hybrid-mixed Element Model Based on Legendre Polynomials for Reissner-Mindlin Plates. *Computer Methods in Applied Mechanics and Engineering, 136*(1-2), 111-126.

Pereira, E.M.B.R., & Freitas, J.A.T. (1996b). A Mixed-Hybrid Finite Element Model based on Orthogonal Functions. *International Journal for Numerical Methods in Engineering, 39*(8), 1295-1312.

Pereira, E.M.B.R., & Freitas, J.A.T. (2000). Numerical Implementation of a Hybrid-Mixed Finite Element Model for Reissner-Mindlin Plates. *Computer & Structures, 74*(3), 323-334.

Pereira, O. J. B. A. (1993). *Um Modelo de Elementos Finitos de Equilíbrio para Elasticidade Tridimensional.* (M.Sc Thesis), Instituto Superior Técnico, Lisbon.

Pereira, O.J.B.A. (1996). *Utilização de Elementos Finitos de Equilíbrio em Refinamento Adaptativo.* (Ph.D Thesis), Instituto Superior Técnico, Lisbon.

Pian, T.H.H. (1964). Derivation of element stiffness matrices by assumed stress distributions. *A.I.A.A. Journal, 2*, 1333–1336.

Pian, T.H.H. , & Tong, P. (1969). Basis of Finite Element Methods for Solid Continua. *International Journal for Numerical Methods in Engineering, 1*, 3-28.

Pina, J.P.P. (2009). *Structural Assessment of Corroded RC Structures Through Numerical Modelling.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Pina, J.P.P., Freitas, J.A.T., & Castro, L.M.S.S. (2004). *Use of Wavelets in Dynamic Analysis*. Paper presented at the Métodos Computacionais em Engenharia, Lisboa.

Piteri, M. A. (1999). *Geracao Automatica de Malhas Hieraquico-Adaptativas em Dominios Bidimensionais e Tridimensionais.* (Ph.D Thesis), Instituto Superior Ténico, Lisbon.

Reddy, J.N. (2004). *An Introduction to Nonlinear Finite Element Analysis*. Oxford: Oxford University press.

Ruoff, G. (1973). Finite Elemente in der Statik. *ch. Die praktische Berechnung der Kombination der Trefftzschen Metode und bei flachen Schalen*, 242-259.

Santos, H.A.F.A. (2009). *Duality in the geometricall exact analysis of frame structures.* (Ph.D Thesis), Instituto Superior Técnico, Lisbon.

Silva, M.C. (2006). *Modelos de Dano em Elementos Finitos Hibridos e Mistos.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Silva, M.C., & Castro, L.M.S.S. (2005). *Modelos Híbridos de Deslocamento com Dano Contínuo.* Paper presented at the Congresso de Métodos Numéricos em Engenharia, Granada.

Silva, M.C., & Castro, L.M.S.S. (2010). *Damage analysis of concrete structures using polynomial wavelets*. Paper presented at the Civil-Comp Press, 2010, Valencia.

Silva, M.J.V. (2002). *Elementos Finitos Híbridos-Mistos de Tensão - Aplicação à análise de barragens abóboda.* (M.Sc Thesis), Instituto Superior Técnico, Lisbon.

Silva, M.V., Barbosa, A.R., Pereira, E.M.B.R., & Castro, L.M.S.S. (2005). *Utilização de um Modelo Híbrido-Misto na Análise Dinâmica de Estruturas Reticuladas Planas.* Paper presented at the Congresso de Metodos Numéricos em Engenharia, Granada.

Stein, E. (1973). Finite Elemente in der Statik. *ch. Die Kombination des modifizierten Treffzschen Verfahrens mit der Methode der Finiten Elemente*, 172-185.

Toma, M. (2009). *Modelling of hydrated soft tissues using hybrid-trefftz finite elements.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Treffz, E. (1926). *Ein Gegenstuck zum Ritzschen Verfahren*. Paper presented at the International Congress Applied Mechanics, Zurich.

Turner, M.J., Clough, R. W, Martin, H.C., & Topp, L.J. (1956). Stiffness and Deflection Analysis of Complex Structures. *Journal of The Aeronautical Sciences, 23*(9), 805-823.

Veubeke, B. M. (1965). *Displacement and Equilibrium models in the Finite Element M ethod*: Stress Analysis: Wiley.

Wang, Z. (2000). *Elastoplastic Structural Analysis with Hybrid Stress Elements.* (Ph.D Thesis), Instituto Superior Técnico, Lisboa.

Washizu, K. (1982). *Variational principles in elasticity and plasticity*.

Zienkiewicz, O.C. (2000). Achievements and some unsolved problems of the finite element method. *International Journal for Numerical Methods in Engineering, 47*, 9-28.

Zienkiewicz, O.C., & Zhu, J.Z. (1992a). A simple error estimator and adaptive procedure for practical engineering analysis. *Computer Methods in Applied Mechanics and Engineering, 24*(1), 337–357.

Zienkiewicz, O.C., & Zhu, J.Z. (1992b). The superconvergent patch recovery (spr) and adaptive finite element refinement. *Computer Methods in Applied Mechanics and Engineering, 101*(1), 207–224.