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# The effects of two temperature and laser pulse on modified couple stress thermoelastic diffusion beam

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**A B S T R A C T**

In this work, we studied the problem of thermoelastic diffusion beams on the basis of modified couple stress theory under the effects of two temperature and laser pulse. The Euler-Bernoulli beam theory and the Laplace transform technique are applied to solve the basic equations of thermoelastic diffusion in the non-dimensional form. The transformed components of displacement, lateral deflection, axial stress, temperature change, concentration, and chemical potential are calculated mathematically to solve the problem. Copper material is used to prepare the mathematical model. The general algorithm of the inverse Laplace transform technique has been calculated numerically. MATLAB software is used to find the results numerically and depict them graphically. The effects of two temperature, laser pulse, and couple stress are presented graphically on the physical quantities. Particular cases are also discussed in the present problem. Laser pulse has many applications in Heat treatment, cutting of plastics, glasses, ceramics, semiconductors and metals, surgery, Lithography, and welding.



## 1. Introduction

The use of couple stresses or moments per unit area acting internally in an elastic continuum found in implicit form was given by (Cauchy, 1851). This paper represented a quite advanced form of the theory, where the energy density is assumed to be a function of gradients of the displacement of any order. (Voigt, 1887) was the first to demonstrate the existence of couple stress in materials. He assumed that the interaction between the two particles of a body through an area element is transmitted not solely by the action of a force vector, but also by a couple stress vector, and the mathematical model for couple stresses was developed by (Cosserat & Cosserat, 1909). (Toupin, 1962) presented the basic concepts and generalizations of classical theory. Several authors studied the couple stress elasticity theory in materials i.e.,(Mindlin & Tiersten, 1962; Koiter, 1964; Green & Rivlin, 1964). (Yang et al., 2002) constructed a new model which is called a modified couple-stress model, in which the couple stress tensor is symmetrical and only one material length parameter is needed to capture the size effect which is caused by microstructure. (Park & Gao, 2006) studied the Bernoulli- Euler beam model based on a modified couple stress theory. Many problems studied by various authors (Reddy & Arvind, 2012; Wang et al., 2015; Akbas, 2017; Ebrahimi & Barati, 2018; Kumar et al., 2019a; Said, 2020; Kumar et al., 2021) in the context of Euler-Bernoulli beam and modified couple stress theories. (Kutbi & Zenkour, 2022) studied the vibration frequency analysis of a microbeam under a temperature pulse. The modified couple stress theory and generalized Lord-Shulman (LS) hyperbolic heat conduction model with a single relaxation time are used to determine the thermoelastic coupled equations for clamped microbeams.

In ultra-short lasers, the pulse duration ranges from nanoseconds to femtoseconds. Two effects play a very important role in ultra-short pulsed laser heating. The first effect is the non-Fourier's heat conduction, which is the modification of the

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Fourier heat conduction theory to account for the effect of thermal relaxation time in the energy carrier's collision process. The second effect is the dissipation of the stress wave due to the coupling between temperature and strain rate, which causes a transformation of mechanical energy associated with the stress wave to the thermal energy of the material. The Euler-Bernoulli beam theory and Laplace transform are used by (Kumar & Devi, 2017a) to discuss the problem of a thermoelastic beam in modified couple stress theory due to laser pulse. Many problems of thermoelasticity under thermal loading due to a laser pulse were studied (Othman & Marin, 2017; Othman & Abd Elaziz, 2019). The wave propagation of generalized micropolar thermoelasticity with three-phase lag models under the effects of memory-dependent derivatives and the influence of thermal loading due to laser pulse was investigated by (Othman & Mondal, 2020).

Diffusion can be defined as a random movement of molecules from a high-concentration region to low concentration region. The study of this phenomenon has many applications in the geophysics and electronics industry. Due to high and low temperatures, the process of heat and mass transfer plays a vital role in many satellite problems, returning spacecraft, and landing on water or land. The concept of thermos-diffusion is used to describe the processes of thermomechanical treatment of metals (Carboning, Nitriding steel, etc.) and these processes are thermally activated, and their diffusing substances being, e.g. Nitrogen, Carbon, etc. In the present day, many oil corporations have used the process of thermos-diffusion to extract oil more efficiently from oil deposits. Thermo-diffusion has many applications in optimal extraction of oil from hydrocarbon reservoirs, fabrication of semiconductor devices in mixtures of metal and molten semiconductors, separation of types like polymers, and the manipulation of the macro-molecules like DNA, etc. (Kumar & Devi, 2017b) studied the modified couple stress thermoelastic diffusion beam under the effects of time by using the eigenvalue approach. (Kumar et al., 2019b) discussed thermoelastic thin beam in modified couple stress with a three-phase lag thermoelastic diffusion model due to thermal and chemical potential sources.

The two temperature theory of heat conduction is formulated by (Chen & Gurtin, 1968). (Chen et al., 1968, 1969) studied the non-simple heat conduction and the thermodynamics of non-simple elastic materials with two temperatures. The normal mode analysis method is used by (Abouelregal & Zenkour, 2017) to solve the problem of two-dimensional micropolar generalized thermoelasticity for a half-space whose surface is traction-free and the conductive temperature at the surface of the half-space. The effects of diffusion and internal heat source were shown (Othman & Said, 2018) on two temperature thermoelastic with a three phase lag model and Green-Naghdi theory without energy dissipation. The normal mode analysis method was used by (Zenkour, 2018) to study the two-temperature multi-phase-lags thermoelasticity theory for the thermomechanical response of microbeams. To capture the size effect in the small-size investigation, the modified couple stress theory is constructed. To show the effects of material length scale parameters on the basis of modified couple stress theory and temperature on the vibration of gold microbeam resonators were investigated by (Hamidi et al., 2019). The modified couple stress and generalized thermoelasticity are applied by (Abouelregal, 2020) to investigate the effects of material length scale parameter on the thermoelastic vibration of a microbeam due to a varying temperature. (Kumar & Kumar, 2021) investigated the thermoelastic vibration of micro and nano-beam resonators with the help of the Euler-Bernoulli beam theory in context with two temperature generalized thermoelasticity theory. They have shown the effects of the two-temperature parameter and micro and nano-beam thickness on the amplitude of deflection, thermal moment, and thermoelastic vibration frequency of the micro and nano-beam resonator graphically. (El-Bary et al., 2022) used hyperbolic two-temperature thermoelasticity theory to study the induced temperature and stress fields in an elastic infinite medium with a cylindrical cavity. The bounding plane surface of the cavity is loaded thermally by a non-Gaussian laser beam with a pulse duration of 2 picoseconds.

The effects of two temperature and laser pulse on modified couple stress thermoelastic diffusion beams are presented in this chapter. The basic equations of thermoelastic diffusion beams are solved by the Euler-Bernoulli beam theory and Laplace transform technique. Numerical results for the displacement, lateral deflection, temperature change, concentration, axial stress, and chemical potential are presented graphically to show the effects of two temperatures, laser pulse, and couple stress. Particular cases of interest are also deduced in the present investigation.

## 2. Basic equations

Following (Yang et al., 2002; Sherief et al., 2004; Chen & Gurtin, 1968; Kumar & Devi, 2017a) the constitutive relations, equations of motion, equation of heat conduction and equation of mass diffusion in a modified couple stress generalized thermoelastic two temperature with diffusive beam in the absence of body forces, body couples, heat and mass diffusive sources are defined as

*2.1 Constitutive equations for couple stress*

$$t_{ij} = \lambda e \delta_{ij} + 2\mu e_{ij} - \frac{1}{2} e_{kij} m_{lk,l} - \beta_1 T \delta_{ij} - \beta_2 C \delta_{ij}, \tag{1}$$

$$m_{ij} = 2\alpha\chi_{ij}, \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right), \tag{3}$$

$$\omega_i = \frac{1}{2} e_{ipq} u_{q,p}, \quad (i, j, k, l = 1, 2, 3). \tag{4}$$

where is the stress tensor, $e_{ij}$ are the components of strain tensor, $e$ is the dilatation, $\delta_{ij}$ is Kronecker's delta, $e_{ijk}$ is the alternate tensor, $\lambda$ and $\mu$ are Lame's constants, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\lambda + 2\mu)\alpha_c$, $\alpha_t, \alpha_c$ are the coefficients of linear thermal expansion and diffusion expansion respectively, $T = \theta_1 - T_0$ is the small temperature increment, $\theta_1$ is the absolute temperature, $T_0$ is the reference temperature assumed to obey the inequality $|T / T_0| << 1$, $C = c_\delta - C_0$ is the change in mass concentration from the initial concentration such that $|C / C_0| << 1$, $c_\delta$ is non-equilibrium concentration and $C_0$ is the mass concentration at natural state, $m_{ij}$ is the couple stress tensor, $\alpha$ is the couple stress parameter, $\chi_{ij}$ is the symmetric curvature tensor, and $\omega_i$ is the rotational vector.

*2.2 Equations of motion for couple stress*

$$\left(\lambda + \mu + \frac{\alpha}{4}\Delta\right)\nabla(\nabla.\boldsymbol{u}) + \left(\mu - \frac{\alpha}{4}\Delta\right)\nabla^2\boldsymbol{u} - \beta_1\nabla T - \beta_2\nabla C = \rho\,\ddot{\boldsymbol{u}}, \tag{5}$$

where $\boldsymbol{u} = (u, v, w)$ is the components of displacement, $\rho$ is the density, $\Delta$ is the Laplacian operator, $\nabla$ is the gradient operator.

*2.3 Equation of heat conduction with two temperature and laser source*

$$K\Delta\phi - \left(\rho c_e \frac{\partial T}{\partial t} + aT_0 \frac{\partial C}{\partial t} + \beta_1 T_0 \frac{\partial}{\partial t}(\nabla.\boldsymbol{u})\right) + S_i = 0, \tag{6}$$

The relation between the conductive temperature and the thermodynamic temperature is given as

$$T = \phi - \xi\nabla^2\phi, \tag{7}$$

where $K^*$ is the thermal conductivity, $c_e$ is the specific heat at constant strain, $\phi$ is the conductive temperature, $\xi > 0$ is a constant called the two-temperature parameter, $S_i$ is the initial heat source.

*2.4 Equation of mass diffusion*

$$D\beta_2\Delta(\nabla.\boldsymbol{u}) + Da\Delta T - Db\Delta C + \frac{\partial C}{\partial t} = 0, \tag{8}$$

where $D$ is the thermoelastic diffusion constant, $a$ is the coefficient describing the measure of thermoelastic diffusion, $b$ is the coefficient describing the measure of mass diffusion.

*2.5 Equation of Chemical Potential*

$$P = -\beta_2 e_{kk} - aT + bC, \tag{9}$$

where $P = p_1 - P_0$ is the change in chemical potential such that $|P / P_0| << 1$, $p_1$ is chemical potential per unit mass at non-equilibrium condition, $P_0$ denotes chemical potential per unit mass at natural state.

## 3 Formulation of the problem

We consider a homogeneous, isotropic, rectangular modified couple stress thermoelastic diffusive beam in a Cartesian coordinate system $O\ xyz$ for the displacement vector $\boldsymbol{u}(x,y,z,t)=(u,v,w)$, and temperature change $T$, and concentration $C$ with dimensions of length $(0\leq x\leq L)$, width $\left(-\frac{d}{2}\leq y\leq\frac{d}{2}\right)$ and thickness $\left(-\frac{h}{2}\leq z\leq\frac{h}{2}\right)$ as shown in Fig. 1. We take the $x-$axis along the length of the beam, and $y-$axis along the width, and $z-$axis along the thickness which also represents the axis of material symmetry. During bending, the plane cross-section initially perpendicular to the axis of the beam remains plane and perpendicular to the neutral surface.

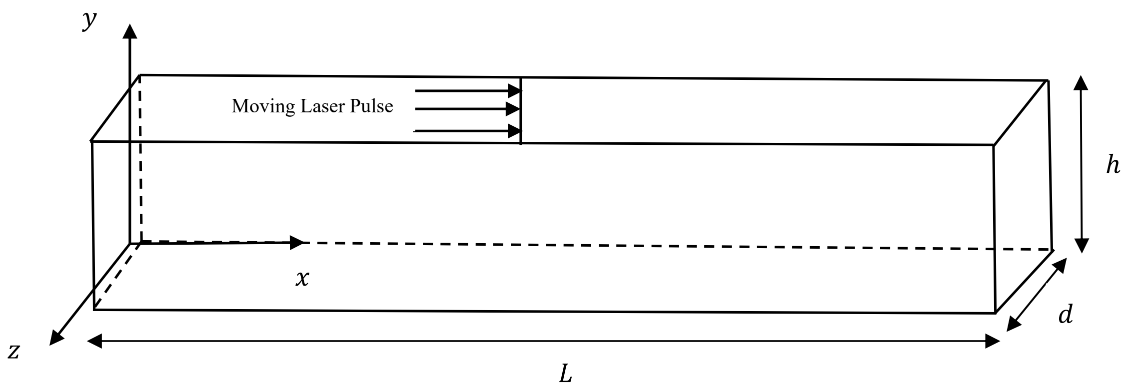


**Fig. 1.** Structure of thermoelastic beam

The displacement components for small deflection of a simple bending problem with Euler-Bernoulli assumption are

$$\left(u\left(x,y,z,t\right),v\left(x,y,z,t\right),w\left(x,y,z,t\right),\phi\left(x,y,z,t\right)\right)=\left(-z\frac{\partial w}{\partial x},0,w\left(x,t\right),\phi\left(x,t\right)\right), \tag{10}$$

Here, $w(x,t)$ denotes lateral deflection of the beam and $t$ is the time.

The bending moment of the cross-section of the beam is given by

$$M_b=M_\sigma+M_m=d\left(\int_{\frac{-h}{2}}^{\frac{h}{2}}t_x z\,dz+\int_{\frac{-h}{2}}^{\frac{h}{2}}m_{xy}dz\right). \tag{11}$$

In Eq. (11), $M_\sigma$ represents the bending moment due to the classic stress and $M_m$ denotes the bending moment due to couple stress tensor. With the help of equation (10), the values of $t_x$, $m_{xy}$, $P$, and $T$ are obtained from Eq. (1), Eq. (2), Eq. (7), and Eq. (9) as

$$t_x=-\left(\lambda+2\mu\right)z\frac{\partial^2 w}{\partial x^2}-\beta_1 T-\beta_2 C, \tag{12}$$

$$m_{xy}=\alpha\frac{\partial^2 w}{\partial x^2}, \tag{13}$$

$$T=\phi-\xi\nabla^2\phi, \tag{14}$$

$$P=\beta_2 z\left(\frac{\partial^2 w}{\partial x^2}\right)-aT+bC. \tag{15}$$

Substituting couple stress Eqs. (2-4), Euler-Bernoulli assumption, axial stress component, and couple stress component from Eq. (10), Eq. (12), and Eq. (13), respectively in Eq. (11), we obtain

$$M_b = -(\lambda + 2\mu) I \frac{\partial^2 w}{\partial x^2} - M_T - M_C - \alpha A \frac{\partial^2 w}{\partial x^2}. \quad (16)$$

Thus $I, M_T$ and $M_C$ are obtained as

$$I = \int_{\frac{-h}{2}}^{\frac{h}{2}} dz^2 dz = \frac{dh^3}{12}, M_T = \beta_1 d \int_{\frac{-h}{2}}^{\frac{h}{2}} Tz\ dz, M_C = \beta_2 d \int_{\frac{-h}{2}}^{\frac{h}{2}} Cz\ dz. \quad (17)$$

where $I$, $M_T$, $M_C$ are the second moment of the cross-section area of the beam, the thermal moment of the beam, and mass moments of the beam respectively. Following (Rao, 2007) the equation of transverse motion of the beam is given as

$$\frac{\partial^2 M_b}{\partial x^2} - \rho A \frac{\partial^2 w}{\partial t^2} = 0. \quad (18)$$

In Eq. (18), $\rho$ is the beam density and $A = dh$ denotes the cross–sectional area of the beam. With the aid of Eq. (16) in Eq. (18), we obtain

$$((\lambda + 2\mu) I + \alpha A) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 M_T}{\partial x^2} + \frac{\partial^2 M_C}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0. \quad (19)$$

With the aid of Eq. (10), the heat conduction Eq. (6), and mass diffusion Eq. (8) can be written as

$$K\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) - \rho c_e \frac{\partial T}{\partial t} - aT_0 \frac{\partial C}{\partial t} + \beta_1 T_0 z \frac{\partial^3 w}{\partial t \partial x^2} + S_i = 0, \quad (20)$$

$$D\beta_2 z \frac{\partial^4 w}{\partial x^4} - Da\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + Db\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2}\right) - \frac{\partial C}{\partial t} = 0. \quad (21)$$

## 4 Solution of the problem

Following Alzahrani and Abbas (2016) assuming the temperature increment and mass concentration variation for a very thin beam in terms of sinusoidal $sin(\chi z)$ function alongwith the thickness of the beam as

$$(T, \phi, C)(x, z, t) = (T_1, \Phi, C_1)(x, t) sin(\chi z), \quad (22)$$

where $\chi = \pi/h$ .

Making use of Eq. (17) and Eq. (22) in Eq. (15) and Eq. (19), we obtain

$$((\lambda + 2\mu) I + \alpha A) \frac{\partial^4 w}{\partial x^4} + \frac{2\beta_1 d}{\chi^2}\left(\frac{\partial^2 T_1}{\partial x^2}\right) + \frac{2\beta_2 d}{\chi^2}\left(\frac{\partial^2 C_1}{\partial x^2}\right) + \rho dh \frac{\partial^2 w}{\partial t^2} = 0, \quad (23)$$

$$T_1 = \Phi - \xi\left(\frac{\partial^2 \Phi}{\partial x^2} - \chi^2 \Phi\right). \quad (24)$$

Multiplying Eq. (20) and Eq. (21) by $z \quad dz$ and integrating over the interval $\left(-h/2, h/2\right)$, and with the aid of Eq. (22), yield the simplified equations as

$$\left(\frac{\partial^2\Phi}{\partial x^2}-\chi^2\Phi\right)-\left(\frac{\rho c_e}{K}\frac{\partial T_1}{\partial t}+\frac{aT_0}{K}\frac{\partial C_1}{\partial t}-\frac{\beta_1 T_0\chi^2}{24K}\frac{\partial^3 w}{\partial t\partial x^2}\right)+\frac{\chi^2}{2K}\int_{-h/2}^{h/2}S_i z dz=0, \tag{25}$$

$$\frac{D\beta_2\pi^2 h}{24}\left(\frac{\partial^4 w}{\partial x^4}\right)-Da\left(\frac{\partial^2 T_1}{\partial x^2}-\chi^2 T_1\right)+Db\left(\frac{\partial^2 C_1}{\partial x^2}-\chi^2 C_1\right)-\frac{\partial C_1}{\partial t}=0, \tag{26}$$

Following Kumar and Devi (2017a) Laser source with a non-Gaussian form temporal profile set along the upper surface $\left(z=\frac{h}{2}\right)$.

$$I(t)=\frac{I_0 t}{t_p^2}exp\left(\frac{-t}{t_p}\right), \tag{27}$$

where $t_p$ is the laser pulse duration, $I_0$ is the laser intensity which is defined as the total energy carried by a laser pulse per unit cross-section of the laser beam. The thermal conduction in the beam can be modeled as a one–dimensional problem with an energy source $S_i$ near the surface, i.e.

$$S_i(z,t)=\frac{(1-R)}{\delta}exp\left(\frac{z-h/2}{\delta}\right)I(t), \tag{28}$$

where $\delta$ is the penetration depth of heating energy and $R$ is the surface reflectivity.

Using Eq. (27) and Eq. (28) in eq. (25), we get

$$\frac{\partial^2\Phi}{\partial x^2}-\chi^2\Phi-\frac{\rho c_e}{K}\frac{\partial T_1}{\partial t}-\frac{aT_0}{K}\frac{\partial C_1}{\partial t}+\frac{\beta_1 T_0\chi^2}{24K}\frac{\partial^3 w}{\partial t\partial x^2}+\frac{\chi^2(1-R)I_0\delta^2 t}{Kht_p^2}e^{-\frac{t}{t_p}}\left[\left(1-\frac{2\delta}{h}\right)+\left(1+\frac{2\delta}{h}\right)e^{-\frac{h}{\delta}}\right]=0. \tag{29}$$

To simplify the problem, we define the following dimensionless quantities:

$$\left(x',z',u',w'\right)=\frac{(x,z,u,w)}{L},\ t'=\frac{vt}{L},\ \left(\Phi',T_1',C_1',t_{xx}'\right)=\frac{\left(\beta_1\Phi,\beta_1 T_1,\beta_2 C_1,t_{xx}\right)}{E},$$
$$\left(M_b',M_T',M_C'\right)=\frac{\left(M_b,M_T,M_C\right)}{dEh^2},\ P'=\frac{P}{bC_0},\ \xi'=\frac{\xi}{L^2},v^2=\frac{E}{\rho}. \tag{30}$$

where $E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$ is Young's modulus, $v=\frac{\lambda}{2(\lambda+\mu)}$ is the Poisson ratio.

Making use of Eq. (30) in Eq. (23), Eq. (24), Eq. (26) and Eq. (29), after surpassing the primes, we obtain

$$\frac{\partial^4 w}{\partial x^4}+a_1\left[\left(\frac{\partial^2 T_1}{\partial x^2}\right)+\left(\frac{\partial^2 C_1}{\partial x^2}\right)\right]+a_2\frac{\partial^2 w}{\partial t^2}=0, \tag{31}$$

$$\frac{\partial^2\Phi}{\partial x^2}-a_3\Phi-\left(a_4\frac{\partial T_1}{\partial t}+a_5\frac{\partial C_1}{\partial t}-a_6\frac{\partial^3 w}{\partial x^2\partial t}\right)+a_7 te^{-\frac{Lt}{vt_p}}=0, \tag{32}$$

$$\left(\frac{\partial^2 C_1}{\partial x^2}-a_3 C_1\right)+a_8\left(\frac{\partial^4 w}{\partial x^4}\right)-a_9\left(\frac{\partial^2 T_1}{\partial x^2}-a_3 T_1\right)-a_{10}\left(\frac{\partial C_1}{\partial t}\right)=0, \tag{33}$$

$$T_1=\Phi-\xi\left(\frac{\partial^2\Phi}{\partial x^2}-a_3\Phi\right), \tag{34}$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9,$ and $a_{10}$ are given in Appendix I.

The Laplace transform is defined by the formula as

$$L\left\{f\left(t\right)\right\}=\int_0^\infty e^{-st}f\left(t\right)dt=\bar{f}\left(s\right). \tag{35}$$

where $s$ is the Laplace transform parameter.

Applying Laplace transform defined by Eq. (35) on Eqs. (31-33), with the aid of Eq. (34) after simplification, we obtain

$$\left(D^4+a_2s^2\right)\bar{w}+a_1D^2\left(1-\xi\left(D^2-a_3\right)\right)\bar{\Phi}+a_1D^2\bar{C}_1=0, \tag{36}$$

$$sa_6D^2\bar{w}+\left(D^2-a_3-sa_4\left(1-\xi\left(D^2-a_3\right)\right)\right)\bar{\Phi}-sa_5\bar{C}_1=-\bar{Q}\left(z,s\right), \tag{37}$$

$$a_8D^4\bar{w}-a_9\left(D^2-a_3\right)\left(1-\xi\left(D^2-a_3\right)\right)\bar{\Phi}+\left(D^2-a_3-a_{10}s\right)\bar{C}_1=0, \tag{38}$$

Here, $Q\left(z,s\right)=a_7\dfrac{v^2t_p^2}{\left(svt_p+L\right)^2}$ is the thermal influence and $D=\dfrac{d}{dx}$.

Solving Eqs. (36-38), we obtain

$$\left(D^8-F_1D^6+F_2D^4-F_3D^2+F_4\right)\begin{bmatrix}\bar{w}\\ \bar{\Phi}\\ \bar{C}_1\end{bmatrix}=\begin{bmatrix}0\\ B_1B_{13}\bar{Q}\\ B_1B_{12}\bar{Q}\end{bmatrix}, \tag{39}$$

where $F_1, F_2, F_3, F_4, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11},$ and $B_{12}$ are given in Appendix II.

The differential equation governing the lateral deflection $\bar{w}$ can take the form

$$\left[\left(D^2-\eta_1^2\right)\left(D^2-\eta_2^2\right)\left(D^2-\eta_3^2\right)\left(D^2-\eta_4^2\right)\right]\bar{w}=0, \tag{40}$$

where $\pm\eta_1, \pm\eta_2, \pm\eta_3$ and $\pm\eta_4$ are the characteristic roots of the equation

$$\eta^8-F_1\eta^6+F_2\eta^4-F_3\eta^2+F_4=0. \tag{41}$$

The solution of lateral deflection, temperature, and concentration is given by

$$\bar{w}\left(x,s\right)=\sum_{i=1}^{4}\left(q_ie^{\lambda_ix}+q_{i+4}e^{-\lambda_ix}\right), \tag{42}$$

$$\bar{\Phi}\left(x,s\right)=\sum_{i=1}^{4}G_iq_ie^{\lambda_ix}+\sum_{i=1}^{4}G_{i+4}q_{i+4}e^{-\lambda_ix}+\delta, \tag{43}$$

$$\bar{C}_1\left(x,s\right)=\sum_{i=1}^{4}H_iq_ie^{\lambda_ix}+\sum_{i=1}^{4}H_{i+4}q_{i+4}e^{-\lambda_ix}+\sum_{i=1}^{8}\varsigma_i, \tag{44}$$

where $q_i$ and $q_{i+4}$, $i=1,2,3,4,$ are constant coefficients and all constants are depending on the Laplace variable $s$ and $\sum_{i=1}^{8}G_i$, $\sum_{i=1}^{8}H_i$, $\sum_{i=1}^{8}\varsigma_i$, $\delta$ are given in Appendix III.

Eq. (10), Eq. (12), and Eq. (14) with the aid of Eq. (22), Eq. (30), and Eq. (35), yield the transformed form of the displacement, axial stress, and chemical potential as

$$\bar{u}\left(x,s\right)=-z\sum_{i=1}^{4}\lambda_i\left(q_ie^{\lambda_ix}-q_{i+5}e^{-\lambda_ix}\right), \tag{45}$$

$$\bar{t}_x = -\frac{(\lambda+2\mu)z}{E}\sum_{i=1}^{4}\left(\lambda_i^2 q_i e^{\lambda_i x} + \lambda_{i+4}^2 q_{i+4} e^{-\lambda_i x}\right) - sin(\chi^* z)\left(\sum_{i=1}^{8}\left((1+\xi a_3)-\xi\lambda_i^2\right)\right)\bar{\Phi} - sin(\chi^* z)\bar{C}_1, \tag{46}$$

$$\bar{P} = \left(\frac{\beta_2 z}{bC_0}\right)\sum_{i=1}^{4}\left(\lambda_i^2 q_i e^{\lambda_i x} + \lambda_{i+4}^2 q_{i+4} e^{-\lambda_i x}\right) - \frac{aEsin(\chi^* z)}{\beta_1 bC_0}\left(\sum_{i=1}^{8}\left((1+\xi a_3)-\xi\lambda_i^2\right)\right)\bar{\Phi} + \frac{Esin(\chi^* z)}{\beta_2 C_0}\bar{C}_1, \tag{47}$$

where

$$\chi^* = \frac{\pi L}{h}.$$

## 5 Boundary conditions

Consider a rectangular beam with both ends are simply supported then the conditions are

$$w(0,t)=0,\ \frac{\partial^2 w(0,t)}{\partial x^2}=0,\ w(\mathrm{L},t)=0,\ \frac{\partial^2 w(L,t)}{\partial x^2}=0. \tag{48}$$

Let us consider the side of beam $x = 0$ is loaded thermally by ramp–type heating, while there is no variation of concentration on it. Thus

$$\Phi(0,t)= g_0\begin{cases}0 & t\le 0,\\ \dfrac{t}{t_0} & 0< t\le t_0,\\ 1 & t>t_0,\end{cases} \tag{49}$$

$$\frac{\partial C(0,t)}{\partial x}=0, \tag{50}$$

where $t_0$ is a non-negative constant called the ramp type parameter and $g_0$ is a constant. We assume that the other side of the beam $x = L$ is thermally insulated and impermeable that the following relation will be satisfied:

$$\frac{d\Phi(L,t)}{dx}=0,\ \frac{dC(L,t)}{dx}=0. \tag{51}$$

With the aid of Eq. (22), Eq. (30), and Eq. (35), in the boundary conditions (48)-(50), we obtain

$$\bar{w}(0,s)=0,\ \frac{d^2\bar{w}(0,s)}{dx^2}=0,\ \bar{\Phi}(0,s)=g_0\left(\frac{1-e^{-st_0}}{t_0 s^2}\right),\frac{d\bar{C}_1(0,s)}{dx}=0, \tag{52}$$

$$\bar{w}(1,s)=0,\ \frac{d^2\bar{w}(1,s)}{dx^2}=0,\ \frac{d\bar{\Phi}(1,s)}{dx}=0,\ \frac{d\bar{C}_1(1,s)}{dx}=0. \tag{53}$$

Substituting the values of lateral deflection, temperature, and concentration from Eqs. (42-44) in the transformed boundary conditions (52) and (53), we obtain the expressions of lateral deflection, conductive temperature, and concentration as

$$\bar{w}(x,s)=\sum_{i=1}^{4}\left(q_i e^{\lambda_i x} + q_{i+4}e^{-\lambda_i x}\right), \tag{54}$$

$$\bar{\Phi}(x,s)=\sum_{i=1}^{4}G_i q_i e^{\lambda_i x} + \sum_{i=1}^{4}G_{i+4}q_{i+4}e^{-\lambda_i x} + \delta_1, \tag{55}$$

$$\bar{C}_1(x,s)=\sum_{i=1}^{4}H_i q_i e^{\lambda_i x} + \sum_{i=1}^{4}H_{i+4}q_{i+4}e^{-\lambda_i x} + \sum_{i=1}^{8}\varsigma_i, \tag{56}$$

The values of displacement $\bar{u}$, axial stress $\bar{t}_x$ and chemical potential $\bar{P}$ are given by

$$\bar{u}(x,s) = -z\sum_{i=1}^{4}\lambda_i\left(q_i e^{\lambda_i x} - q_{i+4}e^{-\lambda_i x}\right), \tag{57}$$

$$\bar{t}_x = -\frac{(\lambda+2\mu)z}{E}\sum_{i=1}^{4}\lambda_i^2\left(p_i e^{\lambda_i x} + q_i e^{-\lambda_i x}\right) - sin(\chi^* z)\left(\sum_{i=1}^{4}\left((1+\xi a_3) - \xi\lambda_i^2\right)\right)\bar{\Phi} - sin(\chi^* z)\bar{C}_1, \tag{58}$$

$$\bar{P} = \left(\frac{\beta_2 z}{bC_0}\right)\sum_{i=1}^{5}\lambda_i^2\left(p_i e^{\lambda_i x} + q_i e^{-\lambda_i x}\right) - \frac{aEsin(\chi^* z)}{\beta_1 bC_0}\left(\sum_{i=1}^{4}\left((1+\xi a_3) - \xi\lambda_i^2\right)\right)\bar{\Phi} + \frac{Esin(\chi^* z)}{\beta_2 C_0}\bar{C}_1, \tag{59}$$

where $\sum_{i=1}^{8} q_i$ are given in Appendix IV.

## 6. Particular Cases

(i) In the absence of couple stress in equations (54)-(59), the results for displacement, lateral deflection, temperature, concentration, chemical potential and axial stress of the beam for thermoelastic diffusion laser two temperature beam are obtained and these results in a special case are similar as obtained by (Abouelregal & Zenkour, 2019) without thermoelastic diffusion.

(ii) In the absence of diffusion variables and two temperature in equations (54)-(59), we find the corresponding expressions for displacement, lateral deflection, temperature, concentration, chemical potential and axial stress in a modified couple stress thermoelastic beam and these results tally with (Kumar & Devi, 2017a).

(iii) If two temperature and laser pulse are zero, in equations (54)-(59), we obtain the corresponding results for modified couple stress thermoelastic diffusion beam.

## 7. Laplace transform Inversion

The solutions of displacement, lateral deflection, temperature, concentration, axial stress and chemical potential are obtained in the Laplace transform domain $(x,s)$. To find the solutions in the transformed domain, use the numerical inversion method. Let $\bar{f}(s)$ be the Laplace transform of a function $f(t)$. The inversion formula of Laplace transform can be written as

$$\bar{f}(s) = L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt, \tag{60}$$

$$f(t) = L^{-1}\left[\bar{f}(s)\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)\,ds, \tag{61}$$

with $s = c + ig$; where $c$ denotes the real part of $\bar{f}(s)$. In Fourier series expansion, we will use the numerical inversion method by which the integral (61) can be written in series as

$$f(t) = \frac{e^{ct}}{t_1}\left[-\frac{1}{2}\mathrm{Re}\bar{f}(c) + \sum_{j=0}^{\infty}\mathrm{Re}\left(\bar{f}\left(c+\frac{ij\pi}{t_1}\right)\right)\cos\left(\frac{j\pi}{t_1}\right) - \sum_{j=0}^{\infty}\mathrm{Im}\left(\bar{f}\left(c+\frac{ij\pi}{t_1}\right)\right)\sin\left(\frac{j\pi}{t_1}\right)\right] - \sum_{j=1}^{\infty}\mathrm{e}^{-2cjt_1}f(2jt_1+t). \tag{62}$$

for $0 \le t \le 2t_1$. The series given in (62) is called the Durbin formula and the last term of this series is called the discretization error. (Honig & Hirdes, 1984) developed a method for accelerating the convergence of the Fourier series and a procedure that computes approximately the best choice of the free parameters.

## 8. Numerical Data

Following (Kumar et al., 2018; Kumar & Devi, 2017a; Zenkour & Abouelregal, 2020) for the numerical data of the physical quantities which are given in the table.

| Symbols of Physical Quantities | Numerical Data | Symbols of Physical Quantities | Numerical Data |
|---|---|---|---|
| $\rho$ | $8.954\times10^{3}$ Kg m$^{-3}$ | $a$ | $1.02\times10^{4}$ m$^{2}$s$^{-2}$K$^{-1}$ |
| $E$ | 120GPa | $b$ | $9\times10^{5}$ Kg$^{-1}$m$^{5}$s$^{-2}$ |
| $\nu$ | 0.34 | $L$ | 1 m |
| $T_0$ | $0.293\times10^{3}$ K | $h$ | 10 m |
| $K$ | $0.386\times10^{3}$ Wm$^{-1}$K$^{-1}$ | $d$ | 1 m |
| $\alpha_t$ | $1.78\times10^{-5}$ K$^{-1}$ | $R$ | 0.93 |
| $\alpha_c$ | $1.98\times10^{-4}$m$^{3}$ Kg$^{-1}$ | $I_0$ | 1000 J/m$^2$ |
| $c_e$ | $0.3831\times10^{3}$J Kg$^{-1}$K$^{-1}$ | $t_p$ | 0.02s |
| $\alpha$ | 2.5 Kgms$^{-2}$ | δ | 0.1 |
| $D$ | $0.85\times10^{-8}$ Kg s m$^{-3}$ | | |

## 9. Figures Discussion

For numerical computations, we used the MATLAB software to find the solutions of displacement, lateral deflection, conductive temperature, concentration, axial stress and chemical potential. In Figs. 2-7, the effects of couple stress one temperature and couple stress two temperatures and Figs. 8-11, the effects of couple stress laser pulse are shown graphically on the resulting quantities respectively.

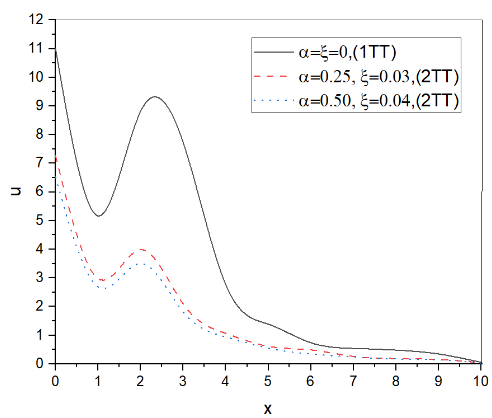


**Fig. 2.** Displacement with length (Two temperature)

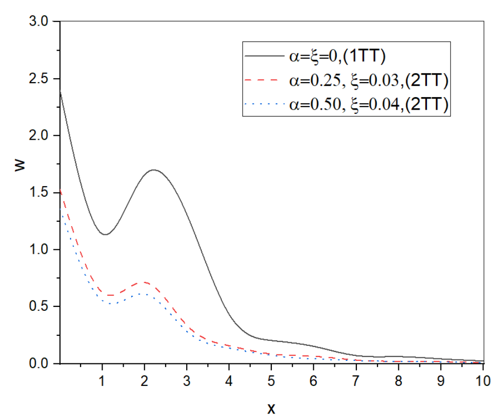


**Fig. 3.** Lateral deflection with length (Two temperature)

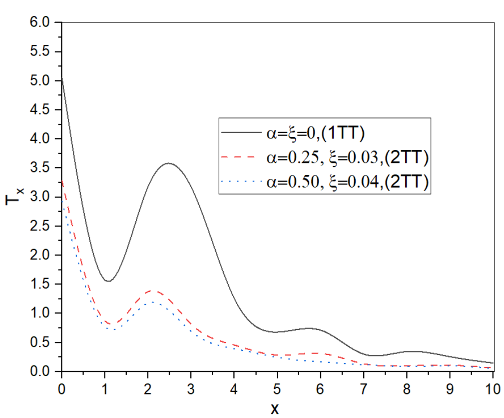


**Fig. 4.** Axial stress with length (Two temperature)

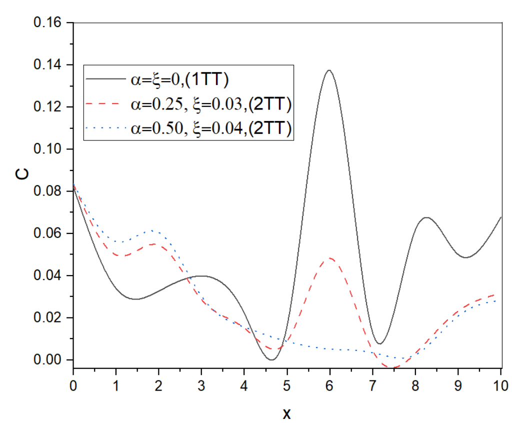


**Fig. 5.** Concentration with length (Two temperature)

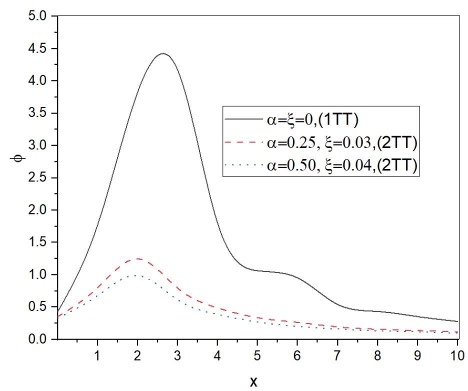


**Fig. 6.** Conductive temperature with length (Two temperature)

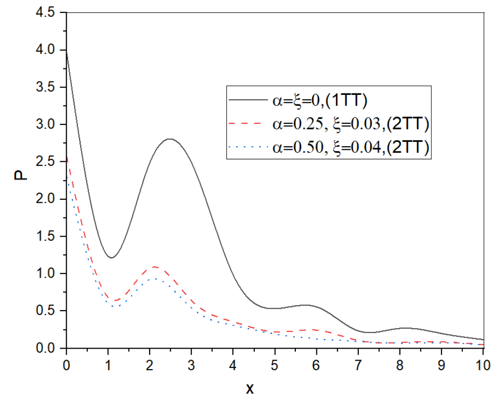


**Fig. 7.** Chemical potential with length (Two temperature)

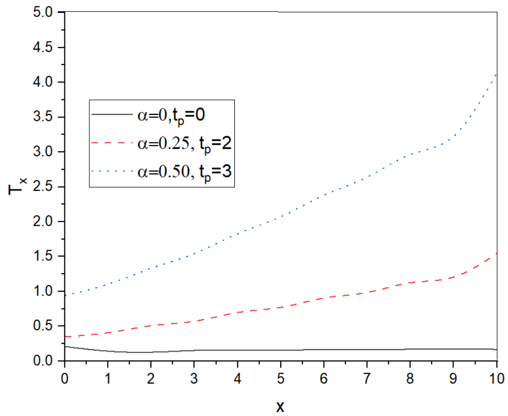


**Fig. 8.** Axial stress with length (Laser Pulse)

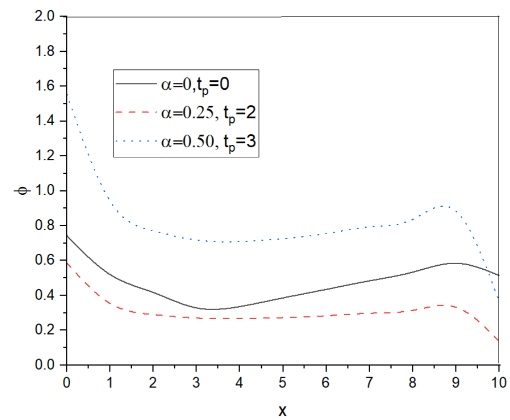


**Fig. 9.** Conductive temperature with length (Laser Pulse)

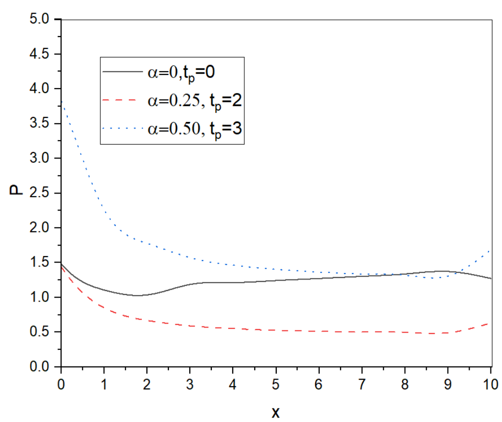


**Fig. 10>** Chemical Potential with length (Laser Pulse)

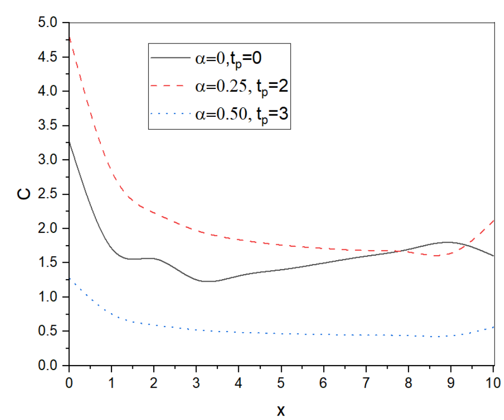


**Fig. 11.** Concentration with length (Laser Pulse)

Figs. (2,3,4), and Fig. 7 show the variation of displacement, lateral deflection, axial stress, and chemical potential with length $x$ along $x$-axis. The behavior of all these figures are almost the same but the variation is different for all the cases of thermoelasticity. In the range , oscillatory nature is shown and in the remaining range the value decreases smoothly. These physical quantities are less effective in the presence of couple stress two temperature thermoelasticity and more effective in the absence of couple stress and one temperature thermoelasticity. Figure 5 depicts the variation of concentration with length $x$ along $x$-axis. Fluctuating nature is shown for concentration in the required region for all the three cases of thermoelasticity. It is observed that the two temperature and couple stress thermoelasticity have more effect on concentration in the range , and less effect in the remaining range but opposite trend is observed for one temperature and without couple stress thermoelasticity. Figure 6 represents the variation of conductive temperature with length $x$ along $x$-axis. The value of conductive temperature increases to attain maxima and then decreases smoothly for all the cases. It is clear from the figure that the conductive temperature has more effect on one temperature and less effect is shown for two temperatures. It is almost the same for couple stress thermoelasticity.

The variation of axial stress on laser pulse along the *x*-axis with length is shown in figure 8. Axial stress plays a more significant role in the presence of two temperatures and couple stress. In figures 9 and 10, Almost same nature is shown for conductive temperature and chemical potential under the effect of laser pulse and couple stress but the trend of variation of these physical quantities are very different. It finds from the study that the variation of both quantities are shown more effective when we increase the value of laser pulse and couple stress. It shows from Fig. 11 that the value of concentration decreases initially and then increases in the remaining range. Concentration has shown higher value for $\alpha = 0.25,\ t_p = 2$ and the smaller value for $\alpha = 0.5,\ t_p = 3$ in comparison with $\alpha = t_p = 0$ .

## 10. Conclusion

This problem deals with the study of thermoelastic diffusion beam in modified couple stress theory in the context of two-temperature generalized thermoelasticity and laser beam with pulse duration is used to heat the surface of half space of the beam. The Euler-Bernoulli beam theory and the Laplace transform are used to solve the governing equations which are in the dimensionless form. To find the solutions of the transformed physical quantities by using the transformed boundary conditions. The numerical technique has been adopted to recover the solutions in the physical domain. Laser pulse has many applications in Industry, Manufacturing, Medical field, Metrology and Communication etc. Effects of one temperature and two temperature as well as laser pulse are shown on the resulting physical quantities. In addition, it is observed that the two temperatures decrease the behavior of all the physical quantities. It is also noticed that the laser pulse plays a vital role in some quantities e.g. Axial stress, Conductive temperature, Concentration, Chemical Potential. But, laser pulse has no effect on lateral deflection and displacement. The present problem is useful for applied Mathematics, Material Science, Industry, and designing of new materials, and it should be beneficial for those who are working in the field of different theories of thermoelastic diffusion beam.

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Appendix I

$$a_1=\frac{2dEL}{\chi^2\left(\left(\lambda+2\mu\right)I+\alpha A\right)},\ a_2=\frac{\rho dh v^2 L^2}{\left(\left(\lambda+2\mu\right)I+\alpha A\right)},\ a_3=\chi^2 L^2,\ a_4=\frac{\rho c_e v L}{K},\ a_5=\frac{aT_0 v L\beta_1}{K\beta_2},$$

$$a_6=\frac{\beta_1^2 T_0\chi^2 v}{24KE},\ a_7=\frac{\beta_1\chi^2 L^3\left(1-R\right)I_0\delta^2}{KEhvt_p^2}\left(\left(1-\frac{2\delta}{h}\right)+\left(1+\frac{2\delta}{h}\right)e^{-\frac{h}{\delta}}\right),\ a_8=\frac{\beta_2^2\pi^2 h}{24bE},\ a_9=\frac{a\beta_2}{b\beta_1},\ a_{10}=\frac{vL}{Db}.$$

Appendix II

$$F_1=\frac{\left(B_7-B_8B_{11}+B_6B_{13}+B_3B_5B_{13}+B_2\left(B_5-B_8B_9\right)-B_4\left(B_7B_9-B_5B_{11}\right)\right)}{\left(B_6-B_8B_{10}+B_3\left(B_5-B_8B_9\right)+B_4\left(B_5B_{10}-B_6B_9\right)\right)},\ F_2=\frac{\left(B_7B_{13}-B_8B_{12}+B_1\left(B_6-B_8B_{10}\right)+B_5\left(B_2B_{13}+B_4B_{12}\right)\right)}{\left(B_6-B_8B_{10}+B_3\left(B_5-B_8B_9\right)+B_4\left(B_5B_{10}-B_6B_9\right)\right)},$$

$$F_3=\frac{B_1\left(B_7+B_6B_{13}-B_8B_{11}\right)}{\left(B_6-B_8B_{10}+B_3\left(B_5-B_8B_9\right)+B_4\left(B_5B_{10}-B_6B_9\right)\right)},\ F_4=\frac{B_1\left(B_7B_{13}-B_8B_{12}\right)}{\left(B_6-B_8B_{10}+B_3\left(B_5-B_8B_9\right)+B_4\left(B_5B_{10}-B_6B_9\right)\right)},$$

$$B_1=a_2s^2,\ B_2=a_1\left(1+\xi a_3\right),\ B_3=\xi a_1,\ B_4=a_1,\ B_5=sa_6,\ B_6=1+s\xi a_4,\ B_7=a_3+sa_4\left(1+\xi a_3\right),\ B_8=-sa_5,\ B_9=a_8,\ B_{10}=\xi a_9,$$

$$B_{11}=a_9\left(1+2\xi a_3\right),\ B_{12}=a_3a_9\left(1+\xi a_3\right),\ B_{13}=\left(a_3+sa_{10}\right).$$

Appendix III

$$\sum_{i=1}^{4}G_i=\sum_{i=1}^{4}\frac{\left(\left(B_8+B_4B_5\right)\lambda_i^2+B_1B_8\right)}{\lambda_i^2\left(\left(B_3B_8-B_4B_6\right)\lambda_i^2+\left(B_4B_7-B_2B_8\right)\right)},\ \sum_{i=5}^{8}G_i=\sum_{i=5}^{8}\frac{\left(\left(B_8+B_4B_5\right)\left(-\lambda_i\right)^2+B_1B_8\right)}{\left(-\lambda_i\right)^2\left(\left(B_3B_8-B_4B_6\right)\left(-\lambda_i\right)^2+\left(B_4B_7-B_2B_8\right)\right)},$$

$$\sum_{i=1}^{4}H_i=\sum_{i=1}^{4}\frac{\left(\left(B_8+B_4B_5\right)\lambda_i^2+B_1B_8\right)\left(\left(B_6+B_3B_5\right)\lambda_i^6-\left(B_2B_5+B_7\right)\lambda_i^4+B_1B_6\lambda_i^2-B_1B_7\right)}{\lambda_i^2\left(\left(B_8+B_4B_5\right)\lambda_i^4+B_1B_8\right)\left(\left(B_3B_8-B_4B_6\right)\lambda_i^2+\left(B_4B_7-B_2B_8\right)\right)},$$

$$\sum_{i=5}^{8}H_i=\sum_{i=5}^{8}\frac{\left(\left(B_8+B_4B_5\right)\left(-\lambda_i\right)^2+B_1B_8\right)\left(\left(B_6+B_3B_5\right)\left(-\lambda_i\right)^6-\left(B_2B_5+B_7\right)\left(-\lambda_i\right)^4+B_1B_6\left(-\lambda_i\right)^2-B_1B_7\right)}{\left(-\lambda_i\right)^2\left(\left(B_8+B_4B_5\right)\left(-\lambda_i\right)^4+B_1B_8\right)\left(\left(B_3B_8-B_4B_6\right)\left(-\lambda_i\right)^2+\left(B_4B_7-B_2B_8\right)\right)},$$

$$\sum_{i=1}^{4}\varsigma_i=\sum_{i=1}^{4}\frac{B_1B_{12}Q\left(\left(B_8+B_4B_5\right)\lambda_i^4+B_1B_8\right)+B_1B_4Q+B_1^2Q\left(B_8B_{12}+B_7B_{13}\right)}{F_4\left(\left(B_8+B_4B_5\right)\lambda_i^4+B_1B_8\right)},$$

$$\sum_{i=5}^{8}\varsigma_i=\sum_{i=5}^{8}\frac{B_1B_{12}Q\left(\left(B_8+B_4B_5\right)\left(-\lambda_i\right)^4+B_8B_1\right)+B_1B_4Q+B_1^2Q\left(B_8B_{12}+B_7B_{13}\right)}{F_4\left(\left(B_8+B_4B_5\right)\left(-\lambda_i\right)^4+B_1B_8\right)},\ \delta=\frac{B_1B_{13}Q}{F_4}.$$

Appendix IV

$$\sum_{i=1}^{4}q_i=\sum_{i=1}^{4}\frac{\Delta_i}{\Delta^*},\ \sum_{i=1}^{4}q_{i+4}=\sum_{i=1}^{4}\frac{\Delta_{i+4}}{\Delta^*},\ \Delta^*=\begin{bmatrix}1&1&1&1&1&1&1&1\\ e^{\lambda_1}&e^{\lambda_2}&e^{\lambda_3}&e^{\lambda_4}&e^{-\lambda_1}&e^{-\lambda_2}&e^{-\lambda_3}&e^{-\lambda_4}\\ \lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2&\lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2\\ \lambda_1^2e^{\lambda_1}&\lambda_2^2e^{\lambda_2}&\lambda_3^2e^{\lambda_3}&\lambda_4^2e^{\lambda_4}&\lambda_1^2e^{-\lambda_1}&\lambda_2^2e^{-\lambda_2}&\lambda_3^2e^{-\lambda_3}&\lambda_4^2e^{-\lambda_4}\\ G_1&G_2&G_3&G_4&G_1&G_2&G_3&G_4\\ \lambda_1G_1e^{\lambda_1}&\lambda_2G_2e^{\lambda_2}&\lambda_3G_3e^{\lambda_3}&\lambda_4G_4e^{\lambda_4}&\lambda_1G_1e^{-\lambda_1}&\lambda_2G_2e^{-\lambda_2}&\lambda_3G_3e^{-\lambda_3}&\lambda_4G_4e^{-\lambda_4}\\ H_1\lambda_1&H_2\lambda_2&H_3\lambda_3&H_4\lambda_4&-H_1\lambda_1&-H_2\lambda_2&-H_3\lambda_3&-H_4\lambda_4\\ H_1\lambda_1e^{\lambda_1}&H_2\lambda_2e^{\lambda_2}&H_3\lambda_3e^{\lambda_3}&H_4\lambda_4e^{\lambda_4}&-H_1\lambda_1e^{-\lambda_1}&-H_2\lambda_2e^{-\lambda_2}&-H_3\lambda_3e^{-\lambda_3}&-H_4\lambda_4e^{-\lambda_4}\end{bmatrix}$$

$\Delta_i\left(i=1,\ldots\ldots\ldots,8\right)$ are obtain by replacing 1st, 2nd, 3rd, 4th, 5th, 6th, 7th and 8th column by $\left[0,0,0,0,\ g_0\left(\frac{1-e^{-st_0}}{t_0s^2}\right)\text{-}\delta_1,0,0,0\right]^{tr}$ in $\Delta_i$.